KAZDAN–WARNER IDENTITIES FOR Q- AND Q'-CURVATURES ON THREE-DIMENSIONAL CR MANIFOLDS

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Abstract. We prove a CR analogue of the Kazdan–Warner identity for the Q-curvature of strictly pseudoconvex three-dimensional CR manifolds. We also prove a similar identity for the Q'-curvature introduced by Case–Yang for the case of three-dimensional locally spherical CR manifolds. Our direct approach to the proofs of our main results reveals an intriguing fact that identities of Kazdan–Warner type also hold for other pseudohermitian invariants which have no Riemannian counterparts.

1. Introduction

The well-known Nirenberg problem asks which smooth functions K on the sphere $S^2 \subset \mathbb{R}^3$ can be realized as the Gauss curvature of a metric g conformal to the standard round metric g_0. If $g = e^{2u} g_0$ has the Gauss curvature K, then u must satisfy the following nonlinear PDE:

$$e^{2u} K = 1 - \Delta u. \tag{1.1}$$

Thus, the Nirenberg problem can be rephrased that for which K this PDE has a solution. An important finding of Kazdan–Warner in [17] about this equation is that, on S^2, a function K for which (1.1) has a solution must satisfy an identity involving the first spherical harmonics. This fact was generalized by Bourguignon–Ezin [1] to an identity involving the scalar curvature and conformal Killing fields on an arbitrary manifold. Precisely, Bourguignon and Ezin proved that on a Riemannian manifold (M, g) with scalar curvature R_g,

$$\int_M X(R_g)\, dV_g = 0, \tag{1.2}$$

holds for each conformal Killing field X on M. Since then, it has been well-understood that many analogous identities hold for various curvature quantities (such as the Q-curvatures [8]) on Riemannian manifolds, providing necessary conditions for the corresponding prescribing curvature problems. In [12], Gover and Ørsted provided universal principles for many identities of this type. We refer the readers to this paper and the references therein for a detailed discussion of Kazdan–Warner type identities in conformal and Riemannian geometries as well as for an extensive list of related references.

In CR geometry, Cheng [6] established the first CR analogue of the Kazdan–Warner identity in the form of Bourguignon–Ezin. Cheng's result involves the Tanaka–Webster

Date: September 3, 2024.
2010 *Mathematics Subject Classification*: Primary 32V20.
Keywords: CR manifolds, Kazdan–Warner identity.

scalar curvature and infinitesimal CR automorphisms. He proved that for a pseudohermitian manifold (M,θ) and an infinitesimal CR automorphism X, it holds that

$$\int_M X(R)\,\theta\wedge(d\theta)^n = 0, \tag{1.3}$$

where R is the Tanaka–Webster scalar curvature. (Cheng proved this identity under some additional conditions. The proof in full generality was given by J. Lee shortly after, see [6].) Cheng's result gives an obstruction for Tanaka–Webster scalar curvature prescription problem on compact CR manifolds, which has been studied extensively, see, e.g., Jerison–Lee [16], Felli–Uguzzoni [10], Gamara–Amri–Guemri [11], and the references therein.

Another Kazdan–Warner-type identity in CR geometry was proved recently for the case of the CR sphere by P.-T. Ho. In [15], Ho studied $\overline{Q}'$-curvature prescription problem, i.e., to prescribe the projection of the Q'-curvature onto the space of CR pluriharmonic functions, and established in [15, Theorem 1.3] a Kazdan–Warner-type identity for $\overline{Q}'$-curvature on the CR sphere.

The purpose of this paper is to prove analogues of Kazdan–Warner identity for the Q- and Q'-curvatures on three-dimensional CR manifolds. On a three-dimensional strictly pseudoconvex CR manifold M with a pseudohermitian structure θ, Q-curvature is defined by Hirachi [13] (see also Feffermann-Hirachi [9])

$$Q = \frac{4}{3}\left(\Delta_b R - 2\operatorname{Im} A_{11,}{}^{11}\right), \tag{1.4}$$

where $A_{11,}{}^{11}$ is the contraction of a second covariant derivative of the pseudohermitian torsion and Δ_b is the nonnegative sub-Laplacian. As proved by Hirachi [13], the Q-curvature defined as above agrees, up to a multiplicative constant, with the boundary value of the log term coefficient of the Szegő kernel and satisfies interesting covariant properties.

The first purpose of this paper is to prove the following Kazdan–Warner identity for the CR Q-curvature.

Theorem 1.1. *Let (M,θ) be a compact three-dimensional strictly pseudoconvex pseudohermitian manifold. If X is an infinitesimal CR automorphism, then*

$$\int_M X(Q)\,\theta\wedge d\theta = 0. \tag{1.5}$$

Theorem 1.1 is analogous to Cheng's theorem for the Tanaka–Webster scalar curvature mentioned above and somewhat similar to Ho's result for the $\overline{Q}'$-curvature in the sphere case. It can also be regarded as a CR counterpart of a result of Delanoë–Robert [8] for the Q-curvature in the conformal geometry, see also Gover–Ørsted [12]. As suggested by a referee, we point out that the CR Q-curvature on a CR manifold M can be obtained from the Q-curvature on the Fefferman's $\mathbb{S}^1$-bundle $\mathbb{S}^1\times M$, as exhibited in Fefferman–Hirachi [9]. Precisely, one has $Q=\pi_* Q_g$, where Q_g is the Q-curvature on the four dimensional manifold $N := \mathbb{S}^1\times M$, $g = g[\theta]$ is the Fefferman metric, and $\pi : \mathbb{S}^1\times M \to M$ is the projection (note that Q_g is $\mathbb{S}^1$-invariant). This suggests that Theorem 1.1 could be deduced from the results of Delanoë–Robert [8] and Fefferman–Hirachi [9]. However, we are not going to further details here.

Integrating the CR Q-curvature on compact manifolds does *not* lead to a nontrivial global invariant as in the conformal counterpart; in three-dimension the total CR Q-curvature is always zero. Moreover, if θ is pseudo-Einstein, then Q-curvature vanishes identically. In this case, Case–Yang [5] introduced the Q'-curvature, together with a Paneitz-type operator for CR pluriharmonic functions denoted by P'. The pair (P', Q') is a CR analogue of the pair of the Paneitz operator P and the Q-curvature on conformal 4-manifolds. We refer the readers to Case–Yang [5] and Case–Gover [4] for a detailed discussion of the Q'-curvature and its associated Paneitz-type operator and their analogies to the conformal counterparts. The second purpose of this paper is to prove the following identity for Q'-curvature on locally *spherical* CR manifolds.

Theorem 1.2. *Let (M, θ) be a compact three-dimensional pseudo-Einstein manifold which is locally CR spherical. If X is an infinitesimal CR automorphism on M, then*

$$\int_M X(Q')\, \theta \wedge d\theta = 0. \tag{1.6}$$

As suggested to the author by an anonymous referee (in 2022), it is expected that this theorem also holds for general pseudo-Einstein manifolds not necessarily being CR spherical. Unfortunately, the author has to leave the general case as an open question for the readers. We would like to stress that the vanishing of the Cartan tensor on CR spherical manifolds is essential to our proof. A direct proof of this theorem in general case could reveal certain symmtries of pseudohermitian invariants which have not been known to us so far.

Although Theorem 1.2 is similar to the result of P.-T. Ho mentioned above, we point out that Theorem 1.3 in Ho [15] is for the projection $\overline{Q}'$ of the Q'-curvature onto the space of CR pluriharmonic functions on the CR sphere and is related to the first spherical harmonics (given by the complex coordinates and their conjugates), while Theorem 1.2 holds for Q'-curvature on arbitrary three-dimensional CR spherical manifolds. On the other hand, our strategy is very different from that used in Ho [15] (and in conformal case in Delanoë–Robert [8]), which uses explicit representations of the CR sphere and a result of Branson–Fontana–Morpurgo [2]. In fact, our proofs are elementary which use an usual integration-by-part argument and various commutation relations in CR geometry given in Lee [18].

Our approach to both theorems above reveals that not only the Q-curvature but two other related quantities also satisfy similar identities. Namely, we found that the sub-Laplacian applied to the scalar curvature as well as the imaginary part of the double divergence of the torsion also satisfy similar identities. This is somewhat intriguing as these two identities, stated explicitly in Lemmas 3.1 and 3.2, seem to have no Riemannian counterparts.

In view of Gover–Ørsted [12], we expect that the identities in Theorems 1.1 and 1.2 follow from certain universal principles in CR geometry. In fact, in [12], Gover–Ørsted generalized Kazdan–Warner identity for arbitrary scalar invariants that are naturally conformally variational. Such invariants are not rare, as pointed out Section 4 of that paper. Gover–Ørsted's result exhibits the fact that various known results regarding generalizations of the Kazdan–Warner identity come from unique principle. However, there could be other Kazdan–Warner-type identities which are not special cases of [12].

For instance, it is interesting to see whether the Riemannian counterpart of Lemma 3.2 holds, i.e., whether one can replace R_g in (1.2) by ΔR_g.

CR analogues of Gover–Ørsted's theorems are definitely interesting and deserve to be treated thoroughly in a separate paper. In the current paper, which is rather short, we focus and confine ourself with integration-by-parts argument which has the merit of being elementary. Moreover, our direct approach reveals somewhat unexpected identities as in Lemmas 3.1 and 3.2 whose existences may not be predicted from an available general principle.

2. Preliminaries

In this section, we briefly recall some basic facts of pseudohermitian geometry. For more details, we refer the readers to Tanaka [19], Webster [20], and Lee [18]. We also briefly introduce the Q-curvature and Q'-curvature in CR geometry; for more details, see Fefferman–Hirachi [9], Hirachi [13, 14], Case–Yang [5], and Case–Gover [4].

Let $(M, T^{1,0}M)$ be a Levi-nondegenerate CR manifold of hypersurface type. If we define $H := \operatorname{Re}\left(T^{1,0}M \oplus T^{0,1}M\right)$, then H is a real two-dimensional sub-bundle of the real tangent bundle TM. If θ is a real 1-form such that $\ker\theta = H$, then the Levi-nondegeneracy of H is equivalent to the everywhere nonvanishing of $\theta\wedge d\theta$. The characteristic (or Reeb) vector field of θ is the unique vector field T such that

$$T \rfloor d\theta = 0, \quad \theta(T) = 1.$$

The real 1-form θ is called a *pseudohermitian structure* and (M,θ) is called a pseudohermitian manifold by Webster [20].

If Z_1 is any local complex vector field spanning $T^{1,0}M$ locally, the admissible 1-form dual to Z_1 is the unique 1-form θ^1 such that the coframe $\{\theta^1, \theta^{\bar 1}, \theta\}$ is dual to the frame $\{Z_1, Z_{\bar 1}, T\}$ and

$$d\theta = i h_{1\bar 1}\theta^1\wedge\theta^{\bar 1} \tag{2.1}$$

for some real function $h_{1\bar 1}$. Here $\theta^{\bar 1} := \overline{\theta^1}$ and $Z_{\bar 1} := \overline{Z_1}$ are the complex conjugations of the respective complex form and vector field. We shall assume that $h_{1\bar 1} > 0$ and say that M is strictly pseudoconvex. The Levi form is denoted by

$$\langle U, \overline{V}\rangle = h_{1\bar 1} U^1 V^{\bar 1} \quad \text{for} \quad U = U^1 Z_1, \overline{V} = V^{\bar 1} Z_{\bar 1}. \tag{2.2}$$

Tanaka [19] and Webster [20] independently introduced a canonical linear connection on the pseudohermitian manifold $(M, T^{1,0}M, \theta)$. It is the connection ∇ on $\mathbb{C}TM$ given in terms of a local holomorphic frame Z_1 by

$$\nabla Z_1 = {\omega_1}^1 \otimes Z_1, \quad \nabla Z_{\bar 1} = {\omega_{\bar 1}}^{\bar 1}\otimes Z_{\bar 1}, \quad \nabla T = 0, \tag{2.3}$$

where the connection form ${\omega_1}^1$ is the complex 1-form uniquely determined by

$$d\theta^1 = \theta^1 \wedge {\omega_1}^1 + {A_{\bar 1}}^1 \theta\wedge\theta^{\bar 1}, \quad {\omega_1}^1 + {\omega_{\bar 1}}^{\bar 1} = d\log h_{1\bar 1}. \tag{2.4}$$

Here, ${A_{\bar 1}}^1$ is the coefficient of the Webster torsion, defined by

$$\operatorname{Tor}(T, Z_{\bar 1}) := \nabla_T Z_{\bar 1} - \nabla_{Z_{\bar 1}} T - [T, Z_{\bar 1}] = {A_{\bar 1}}^1 Z_1.$$

The structure equation for the Tanaka–Webster connection is [20]

$$d{\omega_1}^1 = R h_{1\bar 1}\theta^1\wedge\theta^{\bar 1} + {A_1}^{\bar 1}{}_{,\bar 1}\theta^1\wedge\theta - {A_{\bar 1}}^1{}_{,1}\theta^{\bar 1}\wedge\theta, \tag{2.5}$$

where R is the (real-valued) Webster curvature.

A three-dimensional CR manifold M is locally *CR spherical* if it is locally CR equivalent to the sphere $\mathbb{S}^3$ in $\mathbb{C}^2$. The locally spherity is characterized by the *Cartan umbilical tensor*, a relative invariant of the CR structure on M, introduced by Cartan [3]. Fix a pseudohermitian structure θ, the Cartan umbilical tensor $\mathcal{Q}$ can be interpreted as an endomorphism of the (complexified) holomorphic tangent bundle $\mathbb{C}H$, written locally in Cheng–Lee [7] as

$$\mathcal{Q} = iQ_1{}^{\bar{1}}\theta^1 \otimes Z_{\bar{1}} - iQ_{\bar{1}}{}^1\theta^{\bar{1}} \otimes Z_1. \tag{2.6}$$

Then $\mathcal{Q}$ is a pseudohermitian invariant which is CR-covariant in the sense that if $\hat{\theta} = e^u\theta$ is another pseudohermitian structure, then $\hat{\mathcal{Q}} = e^{-2u}\mathcal{Q}$, as proved in Cheng–Lee [7]. A well-known formula [7, Lemma 2.2] expresses $\mathcal{Q}$ in terms of the covariant derivatives of the scalar curvature and torsion as follows:

$$Q_1{}^{\bar{1}} = \frac{1}{6}R_{,1}{}^{\bar{1}} + \frac{i}{2}RA_1{}^{\bar{1}} - A_1{}^{\bar{1}}{}_{,0} - \frac{2i}{3}A_1{}^{\bar{1}}{}_{,\bar{1}}{}^{\bar{1}}. \tag{2.7}$$

Here, the indices proceeded by a comma indicate covariant derivatives and the 0-index indicates the derivative along the Reeb direction.

A pseudohermitian structure θ on a three-dimensional CR manifold M is said to be *pseudo-Einstein* if

$$R_{,1} - iA_{11,}{}^1 = 0. \tag{2.8}$$

This definition was given in Case–Yang [5], extending the notion of pseudo-Einstein CR manifolds defined by Lee [18] to the three-dimensional case, but the condition (2.8) appeared earlier in Hirachi [13]. An equivalent condition for pseudo-Einsteinian is that the 1-form $\omega_1{}^1 + iR\theta$ is a closed form. The last condition (after an obvious modification) can be used to define the pseudo-Einsteinian in any dimension ≥ 3 (Lee [18]).

By Hirachi [13], when θ is pseudo-Einstein, the Q-curvature vanishes identically. In this case, Case–Yang [5] defined the Q'-curvature. The total Q'-curvature is a secondary invariant, in the sense that it does not change under the change of pseudo-Einstein structures. This curvature quantity, together with its associated operator P', has been studied extensively in recent years, cf. Case–Gover [4] and the references therein. In this paper, we only need the following explicit formula for the Q'-curvature: On a pseudo-Einstein manifold (M, θ), the Q'-curvature is given by

$$Q' = \frac{1}{2}\Delta_b R - |A_{11}|^2 + \frac{1}{4}R^2, \tag{2.9}$$

where R is the Webster scalar curvature, $|A_{11}|^2 = A_{11}A^{11}$ is the squared-norm of the torsion, and Δ_b is the nonnegative sub-Laplacian.

An infinitesimal CR automorphism of M is a real vector field that generates local CR diffeomorphisms. In particular, it is a contact vector field with respect to the contact structure given by the holomorphic tangent space $H(M) := \mathrm{Re}(T^{1,0}M + T^{0,1}M)$. Such a vector field X has the form

$$X = if^{\bar{1}}Z_{\bar{1}} - if^1 Z_1 - fT, \tag{2.10}$$

where $f = -\theta(X)$. It is well-known that the condition that the flow of X preserves the CR structure is equivalent to

$$f_{11} + ifA_{11} = 0. \tag{2.11}$$

See Lee [18]. Notice that the last equation is invariant with respect to the conformal change of pseudohermitian structure.

3. Proofs of Theorems 1.1 and 1.2

Let X be an infinitesimal CR automorphism on M, given in terms of a local holomorphic frame Z_1 as

$$X = if^{\bar{1}} Z_{\bar{1}} - if^1 Z_1 - fT. \tag{3.1}$$

For any function φ on M, integration-by-parts yields

$$\begin{aligned}
\int_M X(\varphi) &= \int_M if^{\bar{1}}\varphi_{\bar{1}} - if^1\varphi_1 - f\varphi_0 \\
&= -\int_M if(\varphi_{\bar{1},}{}^{\bar{1}} - \varphi_{1,}{}^{1}) - f\varphi_0 \\
&= -2\int_M f\varphi_0 \\
&= 2\int_M f_0\varphi,
\end{aligned} \tag{3.2}$$

since $\varphi_{\bar{1},}{}^{\bar{1}} - \varphi_{1,}{}^{1} = -i\varphi_0$ [18, Eq. (2.14)]. We shall use (3.2) in what follows.

Lemma 3.1. *If X is an infinitesimal CR automorphism, then*

$$\int_M X(\operatorname{Im} A_{11,}{}^{11})\,\theta\wedge d\theta = 0. \tag{3.3}$$

Proof. In this proof, we shall use various identities in Lee [18]. First, using the identity $f_{0\bar{1}} = f_{\bar{1}0} + A_{\bar{1}\bar{1}}f^{\bar{1}}$, we compute

$$\begin{aligned}
f_{0\bar{1}\bar{1}} &= (f_{\bar{1}0} + A_{\bar{1}\bar{1}}f^{\bar{1}})_{,\bar{1}} \\
&= f_{\bar{1}0\bar{1}} + A_{\bar{1}\bar{1},\bar{1}}f^{\bar{1}} + A_{\bar{1}\bar{1}}f^{\bar{1}}{}_{,\bar{1}}.
\end{aligned}$$

For the first term on the right hand side, we write

$$f_{\bar{1}0\bar{1}} = f_{\bar{1}\bar{1}0} + A_{\bar{1}\bar{1}}f_{\bar{1},}{}^{\bar{1}} - A_{\bar{1}\bar{1},}{}^{\bar{1}}f_{\bar{1}}$$

we deduce that

$$\begin{aligned}
f_{0\bar{1}\bar{1}} &= f_{\bar{1}\bar{1}0} - (\Delta_b f)A_{\bar{1}\bar{1}} - A_{\bar{1}\bar{1},}{}^{\bar{1}}f_{\bar{1}} + A_{\bar{1}\bar{1},\bar{1}}f^{\bar{1}} \\
&= (ifA_{\bar{1}\bar{1}})_{,0} - (\Delta_b f)A_{\bar{1}\bar{1}} - A_{\bar{1}\bar{1},}{}^{\bar{1}}f_{\bar{1}} + A_{\bar{1}\bar{1},\bar{1}}f^{\bar{1}} \\
&= ifA_{\bar{1}\bar{1},0} + if_0A_{\bar{1}\bar{1}} - (\Delta_b f)A_{\bar{1}\bar{1}} - A_{\bar{1}\bar{1},}{}^{\bar{1}}f_{\bar{1}} + A_{\bar{1}\bar{1},\bar{1}}f^{\bar{1}}.
\end{aligned} \tag{3.4}$$

Here, we used $f_{\bar{1}\bar{1}} = ifA_{\bar{1}\bar{1}}$ which is a consequence of the fact that X is an infinitesimal CR automorphism. Multiplying both sides with A_{11} and taking imaginary parts of both sides, we obtain

$$\begin{aligned}
\operatorname{Im}\int_M (A^{\bar{1}\bar{1}}f_{0\bar{1}\bar{1}}) &= \frac{1}{2}\int_M f|A_{11}|^2_{,0} + \int_M f_0|A_{11}|^2 \\
&\quad - \operatorname{Im}\int_M (f_{\bar{1}}A_{\bar{1}\bar{1},}{}^{\bar{1}}A^{\bar{1}\bar{1}} - f^{\bar{1}}A_{\bar{1}\bar{1},\bar{1}}A^{\bar{1}\bar{1}}).
\end{aligned} \tag{3.5}$$

On the other hand, using the integration-by-part formula (Tanaka [19, section 3.3], Lee [18, Eq. (2.18)]) and the identity $f_{1\bar{1}} - f_{\bar{1}1} = if_0$, we get

$$\begin{aligned}
-\operatorname{Im}\int_M &(f_{\bar{1}} A_{\bar{1}\bar{1},}{}^{\bar{1}} A^{\bar{1}\bar{1}} - f^{\bar{1}} A_{\bar{1}\bar{1},\bar{1}} A^{\bar{1}\bar{1}}) \\
&= \operatorname{Im}\int_M A_{\bar{1}\bar{1}} A^{\bar{1}\bar{1},\bar{1}} f_{\bar{1}} + |A_{11}|^2 f_{\bar{1},}{}^{\bar{1}} + A^{\bar{1}\bar{1}} A_{\bar{1}\bar{1},\bar{1}} f^{\bar{1}} \\
&= \operatorname{Im}\int_M |A_{11}|^2 f_{\bar{1},}{}^{\bar{1}} \\
&= \frac{i}{2}\int_M |A_{11}|^2 \left(f_{1,}{}^{1} - f_{\bar{1},}{}^{\bar{1}}\right) \\
&= -\frac{1}{2}\int_M |A_{11}|^2 f_0.
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5) and applying integration-by-parts, we deduce

$$\begin{aligned}
\int_M X\left(\operatorname{Im} A_{11,}{}^{11}\right) &= 2\int_M f_0 \operatorname{Im}\left(A_{11,}{}^{11}\right) \\
&= 2\operatorname{Im}\int_M A_{11} f_{0,}{}^{11} \\
&= \int_M (f|A_{11}|^2)_{,0} \\
&= 0.
\end{aligned} \tag{3.7}$$

We complete the proof of the lemma. □

Lemma 3.2. *If X is a infinitesimal CR automorphism, then*

$$\int_M X(\Delta_b R)\,\theta \wedge d\theta = 0. \tag{3.8}$$

Proof. From $f_{11} + ifA_{11} = 0$ as above, we have

$$f_{11}A^{11} = -if|A_{11}|^2$$

and thus, as f is of real-valued,

$$f_{11}A^{11} + f_{\bar{1}\bar{1}}A^{\bar{1}\bar{1}} = 2\operatorname{Re}\left(f_{11}A^{11}\right) = 0.$$

On the other hand, using the commutation relations $f_{01} = f_{10} + A_{11}f^1$ and $f_{10\bar{1}} = f_{1\bar{1}0} + f_{11}A^1{}_{\bar{1}} + A^1{}_{\bar{1},1}f_1$ [18, equations (2.14) and (2.15)], we express $f_{01\bar{1}}$ as follows

$$\begin{aligned}
f_{01\bar{1}} &= \left(f_{10} + A_{11}f^1\right)_{,\bar{1}} \\
&= f_{10\bar{1}} + A_{11,\bar{1}}f^1 + A_{11}f^1{}_{,\bar{1}} \\
&= f_{1\bar{1}0} + f_{11}A^1{}_{\bar{1}} + A^1{}_{\bar{1},1}f_1 + A_{11,\bar{1}}f^1 + A_{11}f^1{}_{,\bar{1}} \\
&= f_{1\bar{1}0} + 2\operatorname{Re}\left(f^{\bar{1}}A_{\bar{1}\bar{1},1}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{Re}\int Rf_{01,}{}^{1} &= \operatorname{Re}\int Rf_{1,}{}^{1}{}_{0} + 2\operatorname{Re}\int Rf^{\bar 1}A_{\bar 1\bar 1,}{}^{\bar 1}\\ &= -\operatorname{Re}\int R_{,0}f_{1,}{}^{1} + 2\operatorname{Re}\int Rf^{\bar 1}A_{\bar 1\bar 1,}{}^{\bar 1}. \end{aligned} \tag{3.9}$$

Since $R_{,0} = A_{11,}{}^{11} + A_{\bar 1\bar 1,}{}^{\bar 1\bar 1}$ [18, equation (2.13)], we have

$$\operatorname{Re}\int_M R_{,0}f_{1,}{}^{1} = \operatorname{Re}\int_M A_{11,}{}^{11}f_{1,}{}^{1} + \operatorname{Re}\int_M A_{\bar 1\bar 1,}{}^{\bar 1\bar 1}f_{1,}{}^{1}. \tag{3.10}$$

The first integral on the right-hand side of (3.10) can be transformed as follows:

$$\begin{aligned}\operatorname{Re}\int_M A_{11,}{}^{11}f_{1,}{}^{1} &= -\operatorname{Re}\int_M A_{11,}{}^{1}f_{1,}{}^{11}\\ &= -\operatorname{Re}\int_M A_{11,}{}^{1}\left(f_{\bar 1,}{}^{\bar 1} + if_0\right)^{1}\\ &= -\operatorname{Re}\int_M A_{11,}{}^{1}f_{\bar 1,}{}^{\bar 1 1} - \operatorname{Re}\int_M iA_{11,}{}^{1}f_{0,}{}^{1}\\ &= -\operatorname{Re}\int_M A_{11,}{}^{1}f_{\bar 1,}{}^{\bar 1 1} + \operatorname{Re}\int_M iA_{11}f_{0,}{}^{11} \qquad \text{(use (3.7))}\\ &= -\operatorname{Re}\int_M A_{11,}{}^{1}f_{\bar 1,}{}^{\bar 1 1}. \end{aligned} \tag{3.11}$$

Using [18, equation (2.15)], we compute

$$\begin{aligned} f_{\bar 1,1\bar 1} &= f_{\bar 1,\bar 1 1} + if_{\bar 1,0} - Rf_{\bar 1}\\ &= \left(iA_{\bar 1\bar 1}f\right)_{,1} + if_{0\bar 1} - iA_{\bar 1\bar 1}f_1 - Rf_{\bar 1}\\ &= iA_{\bar 1\bar 1,1}f + if_{0\bar 1} - Rf_{\bar 1}. \end{aligned}$$

Multiplying both sides with $A_{11,\bar 1}$, integrating, and taking the real parts, we obtain

$$\begin{aligned}\operatorname{Re}\int_M A_{11,}{}^{1}f_{\bar 1,1}{}^{1} &= \operatorname{Re}\int_M iA_{11,}{}^{1}f_{0,}{}^{1} - \operatorname{Re}\int_M RA_{11,}{}^{1}f^{1}\\ &= -\operatorname{Re}\int_M iA_{11,}{}^{11}f_0 - \operatorname{Re}\int_M Rf^{1}A_{11,}{}^{1}. \end{aligned}$$

The first term on the right-hand side vanishes, by Lemma 3.1. Combine this with (3.11), we have

$$\operatorname{Re}\int_M A_{11,}{}^{11}f_{1,}{}^{1} = -\operatorname{Re}\int_M A_{11,}{}^{1}f_{\bar 1,1}{}^{1} = \operatorname{Re}\int_M Rf^{1}A_{11,}{}^{1}.$$

On the other hand, we compute

$$\begin{aligned}\operatorname{Re}\int_M A_{\bar 1\bar 1,}{}^{\bar 1\bar 1}f_{1,}{}^{1} &= \operatorname{Re}\int_M A_{\bar 1\bar 1,}{}^{\bar 1\bar 1}\left(f_{\bar 1,}{}^{\bar 1} + if_0\right)\\ &= \operatorname{Re}\int_M A_{\bar 1\bar 1,}{}^{\bar 1\bar 1}f_{\bar 1,}{}^{\bar 1} + \operatorname{Re}\int_M iA_{\bar 1\bar 1,}{}^{\bar 1\bar 1}f_0 \end{aligned}$$

The last term vanishes as proved above. Thus, together with (3.10), we deduce that

$$\int_M R_{,0} f_{1,}{}^{1} = 2\operatorname{Re}\int_M R f^1 A_{11,}{}^{1}.$$

From this, (3.9), and (3.2), we obtain

$$\int_M X(\Delta_b R) = 2\int (\Delta_b R) f_0 = 4\operatorname{Re}\int R f_{01,}{}^{1} = 0, \tag{3.12}$$

which completes the proof of Lemma 3.2. □

Proof of Theorem 1.1. Since the Q-curvature is given by a linear combination of $\Delta_b R$ and $\operatorname{Im}\left(A_{11,}{}^{11}\right)$, Theorem 1.1 follows directly from Lemmas 3.1 and 3.2. □

Now, we move onto proof of Theorem 1.2. We need the following theorem which also gives another pseudohermitian invariant satisfying a Kazdan–Warner identity.

Lemma 3.3. *Put $\Phi = \frac{1}{4}R^2 - |A|^2$. Suppose that θ is pseudo-Einstein and M is locally CR spherical. Then*

$$\int_M X(\Phi)\, \theta \wedge d\theta = 0.$$

Proof. If θ is pseudo-Einstein, then (2.8) holds, that is $R_{,1} = iA_{11,}{}^{1}$. Differentiating this identity, we find that

$$R_{,11} = iA_{11,}{}^{1}{}_{1}. \tag{3.13}$$

Substituting this into the formula for the Cartan tensor Q_{11}, which vanishes identically since M is assumed to be locally CR spherical, we find that

$$0 = Q_{11} = -\frac{1}{2}R_{,11} + \frac{i}{2}RA_{11} - A_{11,0}.$$

Hence,

$$A_{11,0} = \frac{i}{2}\left(RA_{11} - A_{11,}{}^{1}{}_{1}\right). \tag{3.14}$$

On the other hand, applying integration-by-parts twice yields

$$\begin{aligned}\int_M \left(\frac{1}{4}R^2\right)_{,0} f &= \operatorname{Re}\int_M fRA_{\bar{1}\bar{1},}{}^{\bar{1}\bar{1}} \\ &= \operatorname{Re}\int_M A_{\bar{1}\bar{1}}(fR)^{\bar{1}\bar{1}} \\ &= \operatorname{Re}\int_M A_{\bar{1}\bar{1}}\left(f^{\bar{1}\bar{1}}R + 2f^{\bar{1}}R^{\bar{1}} + fR^{\bar{1}\bar{1}}\right).\end{aligned} \tag{3.15}$$

Therefore, substituting the pseudo-Einstein condition (2.8), its consequence (3.13), and the equation of CR infinitesimal automorphisms (2.11) into (3.15), we obtain

$$\begin{aligned}\int_M \left(\frac{1}{4}R^2\right)_{,0} f &= 2\operatorname{Re}\int_M iA^{\bar{1}}{}_{1,}{}^{1}A_{\bar{1}\bar{1}}f^{\bar{1}} + \operatorname{Re}\int_M iA^{\bar{1}\bar{1}}{}_{,}{}^{1}{}_{1}A_{\bar{1}\bar{1}}f \\ &= \operatorname{Re}\int_M iA^{\bar{1}}{}_{1,}{}^{1}A_{\bar{1}\bar{1}}f^{\bar{1}}.\end{aligned} \tag{3.16}$$

Furthermore, from (3.14), we use integration-by-part to get (as R and f are real-valued)

$$\begin{aligned}\int_M |A_{11}|^2_{,0} f &= 2\operatorname{Re}\int_M A^{11}A_{11,0} f \\ &= \operatorname{Re}\int_M iA^{11} f\left(RA_{11} - A_{11,}{}^{1}{}_{1}\right) \\ &= -\operatorname{Re}\int_M if A^{11}A_{11,}{}^{1}{}_{1} \\ &= \operatorname{Re}\int_M if_1 A^{11}A_{11,}{}^{1}. \end{aligned} \tag{3.17}$$

Putting (3.16) and (3.17) together, we find that

$$\int_M \left(\frac{1}{4}R^2 - |A_{11}|^2\right)_{,0} f = 0. \tag{3.18}$$

This complete the proof of the lemma, in view of (3.2). □

Proof of Theorem 1.2. Since the Q'-curvature is a linear combination of $\Delta_b R$ and Φ, Theorem 1.2 follows directly from Lemmas 3.2 and 3.3. □

As in Kazdan–Warner [17], our identities give obstructions for the existence of pseudo-Hermitian structure having prescribed Q-curvature or Q'-curvature. In the example below, we give detailed computations on the sphere giving explicit functions that can not be Q'-curvature of a pseudo-Einstein structure.

Example 1. Consider the sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ defined by the equation

$$\rho(z_1, z_2) := |z_1|^2 + |z_2|^2 - 1 = 0. \tag{3.19}$$

We equip $\mathbb{S}^3$ with its "standard" pseudo-Hermitian structure $\theta = \iota^*(\bar\partial\rho)$, where ι is the inclusion. We put

$$\xi = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} \tag{3.20}$$

and observe, by direct calculations, that $T := i(\xi - \overline{\xi})$ is the Reeb vector field (when being restricted to $\mathbb{S}^3$). Moreover, we put

$$Z_{\bar 1} = z_2\frac{\partial}{\partial \bar z_1} - z_1\frac{\partial}{\partial \bar z_2} \tag{3.21}$$

and see that $Z_{\bar 1}$ forms a frame for $T^{(0,1)}\mathbb{S}^3$. In this frame, the Levi "matrix" is $h_{1\bar 1} = 1$. Further calculations yield

$$\omega_1{}^1 = -2i\theta. \tag{3.22}$$

From this, we can compute various pseudo-Hermitian invariants. In particular, we obtain

$$R = 2, \quad A_{11} = 0, \tag{3.23}$$

and therefore,

$$Q = 0, \quad Q' = 1. \tag{3.24}$$

Take $f = \mathrm{Re}(z_1)$ and $g = -\mathrm{Im}(z_1)$, which are eigenfunctions corresponding to the first positive eigenvalue $\lambda_1 = 1$ of Δ_b on $(\mathbb{S}^3, \theta)$, as can be easily checked. By direct calculations using (3.20), (3.21), and (3.22), we find that

$$f_1 = \frac{1}{2}\bar{z}_2, \quad f_{1,1} = 0, \quad T(f) = g, \quad g_1 = \frac{i}{2}\bar{z}_2, \quad \text{and} \quad T(g) = -f. \tag{3.25}$$

Put $u_\epsilon = 1 + \epsilon g$ ($\epsilon \in \mathbb{R}$) and consider the vector field

$$X = X^{(f)} = if^{\bar{1}} Z_{\bar{1}} - if^1 Z_1 - fT,$$

which is an infinitesimal CR automorphism since $f_{11} = 0$ and $A_{11} = 0$. Using (3.25), we find that

$$\begin{aligned} X(u_\epsilon) = \epsilon X(g) &= \epsilon(if^{\bar{1}} g_{\bar{1}} - if^1 g_1 - fT(g)) \\ &= \epsilon\left(2\,\mathrm{Re}\left(if^{\bar{1}} g_{\bar{1}}\right) + f^2\right) = \epsilon\left(\frac{1}{2}|z_2|^2 + (\mathrm{Re}(z_1))^2\right). \end{aligned}$$

Thus, if $\epsilon > 0$, then $X(u_\epsilon) > 0$ almost everywhere and consequently

$$\int_M X(u)\,\hat{\theta} \wedge d\hat{\theta} > 0$$

for all pseudo-Hermitian structure $\hat{\theta}$. From this, it follows that u_ϵ is the Q'-curvature of some pseudo-Einstein structure on $\mathbb{S}^3$ if and only if $\epsilon = 0$.

For the Q-curvature, we observe that by a similar argument the function $v_\epsilon := \epsilon g$ can not be the Q-curvature of any pseudo-Hermitian structure on $\mathbb{S}^3$ unless $\epsilon = 0$.

Acknowledgment

The author thanks anonymous referees for pointing out a mistake in a previous version of this manuscript and for many useful comments and suggestions.

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