

Classification problems for CR maps from hyperquadrics

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(based on joint works with Hien Nguyen and Michael Reiter)

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The sphere and hyperquadrics

One of the most beautiful models in CR geometry is the sphere

$$\mathbb{S}^{2n+1} = \{Z \in \mathbb{C}^{n+1} \mid \|Z\|^2 = 1\}.$$

For $p \in \mathbb{S}^{2n+1}$, $\mathbb{S}^{2n+1} \setminus \{p\}$ is CR diffeomorphic to the Heisenberg hypersurface

$$\mathbb{H}^{2n+1} = \left\{ \Im(Z_{n+1}) = \sum_{j=1}^n |Z_j|^2 \right\}.$$

Both have a lot of symmetries. Automorphism groups of $\mathbb{S}^{2n+1}$ and $\mathbb{H}^{2n+1}$ have large and finite dimensions, which are known explicitly (Cartan, Tanaka, Chern–Moser). By many reasons, both have a lot of applications in SCV and CR geometry.

The sphere and hyperquadrics

- Local CR maps of $\mathbb{S}^{2n+1}$ extend to global automorphisms (Poincaré, Cartan, Tanaka, Alexander).
- Classification for CR maps from $\mathbb{S}^{2n+1}$ into $\mathbb{S}^{2n'+1}$, $n' > n$ depends on how large n' compared to n .
- “Spherically equivalent classes”: Two maps H and $\tilde{H}$ are equivalent if there are $\psi \in \text{Aut}(\mathbb{S}^{2n'+1})$ and $\gamma \in \text{Aut}(\mathbb{S}^{2n+1})$ such that

$$\tilde{H} = \psi^{-1} \circ H \circ \gamma.$$

- If $n \leq n' \leq 2n - 1$ then all CR maps from $\mathbb{S}^{2n+1}$ into $\mathbb{S}^{2n'+1}$ is equivalent to the linear map $z \mapsto (z, 0)$ (“rigidity”, J. J. Faran 1980s, Webster ($n' = n + 1 \geq 4$)).
- This rigidity fails when $n' \geq 2n + 1$, i.e., there are 1-parameter families of maps (e.g., D’Angelo family).

The sphere and hyperquadrics

- For $n' = 2n$, $n \geq 3$, there are exactly *two* classes, represented by the linear map and the *Whitney map* (Huang–Ji 2001)

$$\mathcal{W}(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, z_1 z_n, z_2 z_n, \dots, z_n^2).$$

- For $n' = 2n + 1$, $n \geq 3$, every rational sphere map is equivalent to one in the D'Angelo's family (Hamada 2005).

$$F_\theta = (z_1, \dots, z_n, \cos(\theta)z_{n+1}, \sin(\theta)z_1 z_{n+1}, \dots, \sin(\theta)z_n z_{n+1})$$

The sphere and hyperquadrics

- For $n' > 2n + 1$, there are further studies (Huang, Ji, Yin, Xu, Lebl, Ebenfelt, Minor, and others) and the “HJY gap conjecture,” which is related to the “SOS conjecture” about the rank of Hermitian sums of squares. The collection of maps becomes “bigger” when the target dimension increases.
- CR maps from $\mathbb{S}^3$ into $\mathbb{S}^5$ are classified into four equivalent classes (Faran 1982)

$$(z_1, z_2) \mapsto \begin{cases} (z_1^2, \sqrt{2}z_1z_2, z_2^2) \\ (z_1, z_2z_1, z_2^2) \\ (z_1^3, \sqrt{3}z_1z_2, z_2^3) \\ (z_1, z_2, 0) . \end{cases}$$

The sphere and hyperquadrics

A real hyperquadric of **signature** $l > 0$ in $\mathbb{C}^{2n+1}$ is given by

$$\Im w = \sum_{j=1}^l |z_j|^2 - \sum_{k=l+1}^n |z_k|^2 = z E_l \bar{z}^t,$$

where E_l is the diagonal matrix with 1's and -1 's appearing on the main diagonal. An interesting model for Levi-nondegenerate ones, Webster's theorem on embedability into real hyperquadric.

Classifications of holomorphic/CR maps between hyperquadrics have been studied extensively (Baouendi, D'Angelo, Huang, Ebenfelt, Zaitsev, Lebl, Xiao, Gao–Ng, ...). The “same signature case”, holomorphic maps $H: \mathbb{H}_l^{2n+1} \rightarrow \mathbb{H}_l^{2N+1}$ exhibits rigidity regardless of the codimension (super rigidity property of Baouendi–Huang in JDG 2005), generalized in various direction by Baouendi, Ebenfelt, Huang, Shroff, and others in later works.

The main interest of this talk is the classification problem for CR maps

$$H: \mathbb{H}_I^{2n+1} \rightarrow \bullet$$

with $I \geq 1, n \geq 2$ and the target $\bullet$ is a “suitable” model: Winkelmann hypersurfaces (Reiter–S. 2024 in IJM), tube over the null cone of a symmetric form (Hien–Reiter–S. 2025, preprint), and a 2-nondegenerate model in $\mathbb{C}^4$ (Reiter–S. 2025, submitted preprint).

Tubes over the null cones of symmetric forms

Consider the null cone of a symmetric form in $\mathbb{R}^4$:

$$\mathcal{C}_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid -x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0\}.$$

The tube $\mathcal{T}_1 = i\mathbb{R} \times \mathcal{C}_1$ in $\mathbb{C}^4$ is given by

$$\rho(z_1, z_2, z_3, w) := -(\Re w)^2 - (\Re z_1)^2 + (\Re z_2)^2 + (\Re z_3)^2 = 0, \quad (1)$$

It is uniformly Levi-degenerate and 2-nondegenerate. A local model defining function for $\mathcal{T}_1$ is

$$\mathcal{X}_1^7 = \left\{ (z, \zeta, w) \in \mathbb{C}^4 \mid \Im w = \frac{|z|_1^2 + \Re(\overline{z z^t} \zeta)}{1 - |\zeta|^2} \right\}, \quad |\zeta| < 1, \quad z = (z_1, z_2),$$

where $|z|_1^2 = z E_1 \bar{z}^t$. Its symmetry algebra has high dimension: $\dim \text{Aut}_0(\mathcal{X}) = 15$.

Tubes over the null cones of symmetric forms

From their defining functions, we can easily construct simple CR maps from the Heisenberg hypersurface $\mathbb{H}_1^5$ into $\mathcal{X}_1^7$:

$$\mathcal{X}_1^7: \Im w = \frac{|z|_1^2 + \Re(\overline{z z^t} \zeta)}{1 - |\zeta|^2}.$$

- $(z, w) \mapsto (0, \zeta, 0)$ sends $\mathbb{H}^3$ into $\mathcal{X}$ (non CR transversally).
- $\ell: (z, w) \mapsto (z, 0, w)$ (the “linear” map).
- Consider $V = \{w\zeta + izz^t = 0\}$. On V , $\Re(\overline{z z^t} \zeta) = -|\zeta|^2 \Im w$. Plugging this into defining function for $\mathcal{X}$, we obtain $\Im w = |z|_1^2$. The map

$$(z, w) \mapsto (z, -izz^t/w, w),$$

singular along $w = 0$, sends an open set of $\mathbb{H}_1^5$ into $\mathcal{X}_1^7$.

These are similar to the case of Heisenberg hypersurface and tube over the future light cone ($I = 0$) (Reiter–S. 2022). Q: *What are the other maps?*

Let M and M' be CR manifolds and let $H: (M, p) \rightarrow (M', p')$ a germ of smooth CR maps. The product group

$$G := \text{Aut}(M, p) \times \text{Aut}(M', p')$$

acts on the space of map germs.

Definition

We say that H and $\tilde{H}$ are *equivalent* if there exist germs of local CR diffeomorphisms $\gamma: (M, p) \rightarrow (M, \tilde{p})$ and $\psi: (M', \tilde{H}(\tilde{p})) \rightarrow (M', H(p))$, such that

$$H = \psi \circ \tilde{H} \circ \gamma^{-1}.$$

Previous results: The case of Heisenberg hypersurface

Theorem (Reiter–S., arXiv 2024)

Let $U \subset \mathbb{H}^5$ and $H: U \rightarrow \mathcal{X} \subset \mathbb{C}^4$ a C^2 -smooth CR map.

(a) If H is CR transversal at a point, then it is transversal on U . The germs (H, q) , $q \in U$, are mutually equivalent and are equivalent to exactly one of the germs at the origin of the following maps:

(i) $\ell(z_1, z_2, w) = (z_1, z_2, 0, w),$

(ii) $r(z_1, z_2, w) = \left(\frac{z_1(1+iw)}{1-w^2}, \frac{z_2(1-iw)}{1-w^2}, \frac{2(z_1^2 - z_2^2)}{1-w^2}, \frac{w}{1-w^2} \right),$

(iii) $\iota(z_1, z_2, w) = \frac{2}{1 + \sqrt{1 - 4w^2 - 4i(z_1^2 + z_2^2)}} (z_1, z_2, w, w).$

(b) H is nowhere CR transversal and equivalent to a map $(z_1, z_2, w) \mapsto (0, 0, \phi(z_1, z_2, w), 0)$ for a C^2 -smooth CR function ϕ with $\phi(0) = 0$.

The case of hyperquadrics

Theorem (Hien–Reiter–S. 2025, in preparation)

Let $n \geq 3$, $1 \leq l < n - 1$ and U be an open subset of $\mathbb{H}_l^{2n-1} \subset \mathbb{C}^n$. Let $H: U \rightarrow \mathcal{X}_l^{2n+1} \subset \mathbb{C}^{n+1}$ be a transversal CR map. Then

- ① If $n = 3$ and $l = 1$, then H is equivalent to exactly one of the germs at the origin of the following maps:

- ① $H_j(z, w) = \left(\frac{z + iwzP_j}{1 + \varepsilon_j w^2}, \frac{2zP_j z^t}{1 + \varepsilon_j w^2}, \frac{w}{1 + \varepsilon_j w^2} \right)$, where $z = (z_1, z_2)$, z^t is its transpose, and P_j is one of the following five 2×2 matrices

$$P_1 = 0, \quad P_2 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and $\varepsilon_j = \det(P_j) \in \{-1, 0, 1\}$ is the determinant of P_j .

- ② ...
...

The case of hyperquadrics

Theorem (Hien–Reiter–S. 2025, in preparation)

Let $n \geq 3$, $1 \leq l < n - 1$ and U be an open subset of $\mathbb{H}_l^{2n-1} \subset \mathbb{C}^n$. Let $H: U \rightarrow \mathcal{X}_l^{2n+1} \subset \mathbb{C}^{n+1}$ be a transversal CR map. Then

① If $n = 3$ and $l = 1$, then H is equivalent to exactly one of the germs at the origin of the following maps:

① ...

②
$$l(z, w) = \frac{2(z, w, w)}{1 + \sqrt{1 - 4i(zz^t - iw^2)}}.$$

② If $n \geq 4$, then H is equivalent to exactly one of the germs at the origin of the following maps:

① $\ell^{(n)}(z, w) = (z, 0, w),$

②
$$l^{(n)}(z, w) = \frac{2(z, w, w)}{1 + \sqrt{1 - 4i(zz^t - iw^2)}}, \text{ where } z = (z_1, z_2, \dots, z_{n-1}) \text{ and } z^t \text{ is its transpose.}$$

The case of hyperquadrics

Theorem (Hien–Reiter–S., 2025 (in preparation))

Let H be a rational holomorphic map that sends the germ of hyperquadric $\mathbb{H}_1^5$ into $\mathcal{X}_1^7$. Then H is given by

$$H(z, w) = H_A(z, w) = \left(\frac{4z + 2iwzAz^t}{4 + |A|w^2}, \frac{4zAz^t}{4 + |A|w^2}, \frac{4w}{4 + |A|w^2} \right), \quad (2)$$

where

$$A = \begin{pmatrix} \alpha & i\beta \\ i\beta & -\alpha \end{pmatrix}, \quad (3)$$

with $\alpha, \beta \in \mathbb{R}$, so that $|A| = -\alpha^2 + \beta^2$. Conversely, each matrix A of the form (3) gives rise to a holomorphic map sending the hyperquadric into $\mathcal{X}_1^7$.

Another interesting model

A 2-nond. model in $\mathbb{C}^4$ whose stability group has the “submaximal dimension” 16:

$$\mathcal{M} = \{(z, \zeta, w) \in \mathbb{C}^4 \mid \Im(w + z_1^2 \bar{\zeta}) = z_1 \bar{z}_2 + \bar{z}_1 z_2\}, \quad (4)$$

where $z = (z_1, z_2)$ (as a row vector). This model appeared in a paper by Labovskii in 1997.

Fact: $\mathcal{M} \sim \widetilde{\mathcal{M}}$ in $\mathbb{C}^4$ defined by

$$\widetilde{\rho}(z_1, z_2, \zeta, w) = \Im\left(w + \frac{1}{2}\bar{\zeta}(z_1 + z_2)^2\right) - |z_1|^2 + |z_2|^2, \quad (5)$$

It is easy to see that the linear map $(z, w) \mapsto (z, 0, w)$ is a transversal CR map from $\mathbb{H}_1^5$ into $\widetilde{\mathcal{M}}$. *Our question is what about other maps?*

Theorem (Reiter–S. 2025 (submitted))

Assume that (H, p) is a germ at $p \in \mathbb{H}_1^5$ of rational holomorphic maps sending $(\mathbb{H}_1^5, p)$ into $\widetilde{\mathcal{M}}$, then (H, p) is equivalent to one of the following five mutually inequivalent germs at the origin represented by

$$\ell(z, w) = (z, 0, w), \quad (6)$$

$$P_\varepsilon(z, w) = \left(z + \frac{i}{2}\varepsilon wzA, 2i\varepsilon(z_1^2 - z_2^2), w \right), \quad (7)$$

$$R_\varepsilon(z, w) = \left(\frac{z + \varepsilon wzP}{(1 - \varepsilon w)^2}, \frac{8\varepsilon z_1 z_2}{(1 - \varepsilon w)^2}, \frac{w - 2\varepsilon w^2}{(1 - \varepsilon w)^2} \right), \quad (8)$$

where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad z = (z_1, z_2), \quad \varepsilon \in \{-1, 1\}. \quad (9)$$

Step 1. A partial normal form:

Let $H: (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^4, 0)$ be a holomorphic map sending the germ $(\mathbb{H}_1^5, 0)$ into $(\widetilde{\mathcal{M}}, 0)$. By composing with CR automorphisms fixing the origin of the source and target γ and φ , whose explicit forms are well-known as above, one obtained a new map $\widetilde{H} = \varphi^{-1} \circ H \circ \gamma$ in the following “partial normal form”

$$f(z, w) = z + \frac{i}{2}wzA + O(2), \quad (10)$$

$$\phi(z, w) = \lambda w + zBz^t + O(3), \quad (11)$$

$$g(z, w) = w + O(3), \quad (12)$$

with $z = (z_1, z_2)$, $f = (f_1, f_2)$, $A, B \in \text{Mat}(2; \mathbb{C})$, and $B = [b_{jk}]$ is symmetric with b_{11} is purely imaginary.

All that needed are explicit formulas for the automorphisms.

Step 2: Determine the map from partial normal form

Theorem (Reiter–S. 2025)

Let H be a holomorphic rational map defined in a neighborhood U of the origin in $\mathbb{C}^3$ sending $U \cap \mathbb{H}_1^5$ into $\mathcal{M}$. Assume that H has the partial normal form as in (10), (11), and (12). Then

$$A = \begin{bmatrix} \alpha & -\alpha - i\beta \\ \alpha - i\beta & -\alpha \end{bmatrix}, \quad B = 2 \begin{bmatrix} i\alpha & \beta \\ \beta & -i\alpha \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad (13)$$

and

$$H_{\alpha,\beta}(z, w) = \left(\frac{4z + 2iwz(A + 2i\beta I_2)}{(2 - \beta w)^2}, \frac{4zBz^t}{(2 - \beta w)^2}, \frac{4w - 4\beta w^2}{(2 - \beta w)^2} \right). \quad (14)$$

Step 2.1 Determine $H(z, 0)$ (H along the first Segre set)

There is a real-valued function $Q = Q(z, w, \bar{z}, \bar{w})$ which is defined in a neighborhood of the origin such that

$$g - \bar{g} - 2if_1\bar{f}_1 + 2if_2\bar{f}_2 + \frac{1}{2}\Upsilon^2\bar{\phi} - \frac{1}{2}\bar{\Upsilon}^2\phi = Q(w - \bar{w} - 2iz_1\bar{z}_1 + 2iz_2\bar{z}_2), \quad (15)$$

where $\Upsilon := f_1 + f_2$. Setting $w = \bar{w} + 2i(|z_1|^2 - |z_2|^2)$, we obtain the following form of mapping equation:

$$\begin{aligned} &g(z, w) - \bar{g}(\bar{z}, \bar{w}) - 2if_1(z, w)\bar{f}_1(\bar{z}, \bar{w}) + 2if_2(z, w)\bar{f}_2(\bar{z}, \bar{w}) \\ &+ \frac{1}{2}\Upsilon(z, w)^2\bar{\phi}(\bar{z}, \bar{w}) - \frac{1}{2}\bar{\Upsilon}(\bar{z}, \bar{w})^2\phi(z, w) \Big|_{w=\bar{w}+2i(|z_1|^2-|z_2|^2)} = 0. \end{aligned} \quad (16)$$

Setting $\bar{z}_1 = \bar{z}_2 = \bar{w} = 0$ in (16), we obtain

$$g(z, 0) = 0. \quad (17)$$

Step 2.1 Determine $H(z, 0)$ (H along the first Segre set)

We will frequently differentiate the mapping equation along the following tangential CR vector fields

$$L_1 = \frac{\partial}{\partial \bar{z}_1} - 2iz_1 \frac{\partial}{\partial \bar{w}}, \quad L_2 = \frac{\partial}{\partial \bar{z}_2} + 2iz_2 \frac{\partial}{\partial \bar{w}}. \quad (18)$$

Lemma

If $\Upsilon = f_1 + f_2$, then $\Upsilon(z, 0)$ satisfies the following quadratic relation

$$\bar{\lambda}(z_1 + z_2)\Upsilon(z, 0)^2 + 2\Upsilon(z, 0) - 2(z_1 + z_2) = 0. \quad (19)$$

Consequently, if H is rational, then $\lambda = 0$ and $f(z, 0) = z$.

Step 2.1 Determine $H(z, 0)$ (H along the first Segre set)

Proof.

Applying the tangential CR vector field $X = L_2 - L_1$ to the mapping equation (16) and evaluating at $\bar{z}_1 = \bar{z}_2 = \bar{w} = 0$ we obtain (19), after some simplification. The last statement then follows by noting that the discriminant of the quadratic relation (19) is

$$\Delta = 4(1 + 2\bar{\lambda}(z_1 + z_2)^2), \quad (20)$$

which is a square of a rational function of z_1 and z_2 if and only if $\lambda = 0$. With this, one can determine $f(z, 0)$ by applying L_1 and L_2 to the mapping equation (16) and setting $w = \bar{w} = 0$. The proof is complete. □

Proof

In the next step, we determine $\phi(z, 0)$, which will be a polynomial of total degree at most 2 of z_1 and z_2 .

Lemma

Let H be a holomorphic rational map defined in a neighborhood U of the origin in $\mathbb{C}^3$ sending $U \cap \mathbb{H}_1^5$ into $\mathcal{M}$. Assume that H has the partial normal form as in (10), (11), and (12). Then

$$A = \begin{bmatrix} \alpha & -\alpha - i\beta \\ \alpha - i\beta & -\alpha \end{bmatrix}, \quad B = 2 \begin{bmatrix} i\alpha & \beta \\ \beta & -i\alpha \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad (21)$$

Moreover,

$$\phi(z, 0) = zBz^t. \quad (22)$$

Proof.

Skipped (similar, more computations, using Huang's Lemma). □

From $H(z, 0) = (z, 0, w)$, we obtain several holomorphic equations:

$$2i\alpha w^2 \Upsilon^2 - 8wz_1 f_1 + 8z_1^2 g + w^2 \phi = 0, \quad (23)$$

$$2i\alpha w^2 \Upsilon^2 + 8wz_2 f_2 - 8z_2^2 g - w^2 \phi = 0, \quad (24)$$

$$2w(i\alpha z_2 - \beta z_1) \Upsilon^2 - 4z_1 z_2 f_1 + 4z_1^2 f_2 + w(z_1 + z_2) \phi = 0, \quad (25)$$

where $\Upsilon = f_1 + f_2$ for short.

Proof.

setting $\bar{w} = 0$ in the mapping equation (16), we obtain an identity of the form

$$\mathcal{P}_0(z, \bar{z}, \bar{H}(\bar{z}, 0), H(z, 2i(z_1 \bar{z}_1 - z_2 \bar{z}_2))) = 0, \quad (26)$$

where $\mathcal{P}_0$ is a polynomial in its arguments, which is explicit and depend on the parameters α and β . □

Proof.

Next, plugging in the formula for $\overline{H}(\bar{z}, 0)$, we obtain

$$\tilde{\mathcal{P}}_0(z, \bar{z}, H(z, 2i(z_1\bar{z}_1 - z_2\bar{z}_2))) = 0, \quad (27)$$

where $\tilde{\mathcal{P}}_0$ is also an explicit polynomial. Further, setting $\bar{z}_2 = 0$ and $\bar{z}_1 = \frac{w}{2iz_1}$, we obtain a rational relation of the form

$$\mathcal{R}_0(z, \bar{z}_1, H(z, w)) = 0. \quad (28)$$

Setting $\bar{z}_1 = 0$ in (28) we obtain the first holomorphic equation (23). Differentiating in $\bar{z}_1$ once and setting $\bar{z}_1 = 0$ we obtain (24), and differentiating twice in $\bar{z}_1$ and setting $\bar{z}_1 = 0$, we obtain the third one (25).

Viewed as a system of linear equations in f_1, f_2, ϕ , they can be solved easily. □

Lemma

Put

$$\psi = \left(\frac{f_1 + f_2}{z_1 + z_2} \right)^2, \quad \eta = \frac{g}{w}. \quad (29)$$

Then

$$f = \eta z + \frac{i}{2} \psi w z A, \quad (30)$$

$$\phi = \psi z B z^t. \quad (31)$$

This partially determine $H(z, w)$. The next step we need to determine $H_w(z, 0)$ and invoke the “reflection principle” to obtain one more holomorphic equation.

Proof (continued)

Lemma

Put

$$\psi = \left(\frac{f_1 + f_2}{z_1 + z_2} \right)^2, \quad \eta = \frac{g}{w}. \quad (32)$$

Then

$$f = \eta z + \frac{i}{2} \psi w z A, \quad (33)$$

$$\phi = \psi z B z^t. \quad (34)$$

Lemma

With $\Upsilon = f_1 + f_2$ as above, it holds that

$$2\alpha w(\beta w - 2)\Upsilon^2 + 4z_1(\alpha w - 2i)f_1 + 4wz_1(\alpha - i\beta)f_2 + iw(2 - \beta w)\phi + 8iz_1^2 = 0. \quad (35)$$

Solve the above system:

$$A = \begin{bmatrix} \alpha & -\alpha - i\beta \\ \alpha - i\beta & -\alpha \end{bmatrix}, \quad B = 2 \begin{bmatrix} i\alpha & \beta \\ \beta & -i\alpha \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad (36)$$

and

$$H_{\alpha,\beta}(z, w) = \left(\frac{4z + 2iwz(A + 2i\beta I_2)}{(2 - \beta w)^2}, \frac{4zBz^t}{(2 - \beta w)^2}, \frac{4w - 4\beta w^2}{(2 - \beta w)^2} \right). \quad (37)$$

Note: The strategy of this proof was used recently by Reiter–S. (from ball to type IV domain), Della Sala–Lamel–Reiter–S. (highly degenerate sphere maps).

Thank you!