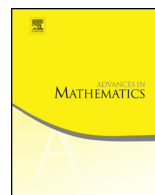Contents lists available at ScienceDirect

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On CR maps from the sphere into the tube over the future light cone

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A R T I C L E   I N F O

A B S T R A C T

We determine all local smooth or formal CR maps from the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ into the tube $\mathcal{T} := \mathcal{C} \times i\mathbb{R}^3 \subset \mathbb{C}^3$ over the future light cone $\mathcal{C} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2, \ x_3 > 0\}$. This result leads to a complete classification of proper holomorphic maps from the unit ball in $\mathbb{C}^2$ into Cartan's classical domain of type IV in $\mathbb{C}^3$ that extend smoothly to some boundary point. Up to composing with CR automorphisms of the source and target, the classification consists of four algebraic maps. Two maps among them were known earlier in the literature. They can be generalized to higher dimensional cases and were shown to be "rigid" when the source dimension is at least 4 in a recent paper by Xiao and Yuan. Two newly discovered quadratic polynomial maps provide counterexamples to a conjecture appeared in the same paper for the case of dimension two.

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1. Introduction

The study of proper holomorphic maps between balls goes back to Poincaré [41] and Alexander [2] who showed that the proper holomorphic self-maps of a ball in $\mathbb{C}^N$, $N \geqslant 2$, are precisely the automorphisms which have explicit parametrizations. When the target is a ball of higher dimension, the discovery of inner functions implies that there are proper holomorphic maps between balls that do not extend to any boundary point [1,17,35,36]. On the other hand, by a well-known result of Forstnerič [15], proper holomorphic maps between balls, which extend sufficiently smooth to a boundary point, extend to rational maps. Moreover, by Cima–Suffridge [8] the rational maps have no poles on the boundary and hence induce global CR maps between spheres.

There is an interesting phenomenon for CR maps of spheres (i.e. *sphere maps*) depending on the size of the *codimension* (i.e. the difference of the CR dimensions of the source and target sphere). When the codimension is "low," the maps are "rigid," which means that there is only the linear map up to composition with automorphisms, see [45,13,20]. When the codimension is not too large, the sphere maps may be completely classified [12,13,22–24]. In case there is no restriction on the codimension, the collection of sphere maps is known to be huge [9,18]. A particular interesting case is when the source is $\mathbb{S}^3 \subset \mathbb{C}^2$. In this case, the smooth CR maps into $\mathbb{S}^5 \subset \mathbb{C}^3$ have been classified by Faran [12] as four "spherical equivalence" classes of maps. See also [7,26,42] for different proofs. However, a classification of the maps from $\mathbb{S}^3$ into $\mathbb{S}^7$ is only available for monomial maps [9].

Besides the study of proper holomorphic maps of balls, there is a large literature devoted to the study of proper holomorphic maps between various models such as the complex ellipsoids, Reinhardt domains, and the classical domains. In particular, there are many works devoted to the study of proper holomorphic maps as well as isometric holomorphic embeddings with respect to the Bergman metrics from one bounded symmetric domain into another by Mok, Ng, Kim–Zaitsev, Xiao, Yuan and others, see, e.g. [40,28,47,46]. In [47], Xiao–Yuan give complete classifications of proper maps from $\mathbb{B}^n$ into $D_m^{\mathrm{IV}} \subset \mathbb{C}^m$ (the Cartan's classical domain of type IV of m dimension) in the "low" codimensional case (i.e., $4 \leqslant n \leqslant m - 1 \leqslant 2n - 4$) and for the holomorphic isometries from $\mathbb{B}^n$ into $D_{n+1}^{\mathrm{IV}} \subset \mathbb{C}^{n+1}$ for all $n \geqslant 2$. When the codimension is higher, there are many more explicit examples of proper holomorphic maps given in [47]. However, in the case $n = 2$ and $m = 3$, only two equivalence classes of proper holomorphic maps were known (and they are all isometries), yet the classification problem was left open in these particular dimensions. We refer the reader to, e.g., [38–40,47] and their references for more detailed information.

In this paper, we study local CR maps from $\mathbb{S}^3$ into the well-known *tube over the future light cone* $\mathcal{T}$ of real dimension 5 which has been of interest in many papers, e.g., [11,27]. This is an everywhere Levi-degenerate 2-nondegenerate CR manifold (cf. [11, Example 4.2.1]) which is homogeneous and has a "large" stability group; see, e.g., [27,14]. Moreover, $\mathcal{T}$ is locally CR equivalent to the smooth boundary part $\mathcal{R}$ of D_3^{IV}

and our study provides a complete classification of proper holomorphic maps from $\mathbb{B}^2$ into D_3^{IV} which extend smoothly to a boundary point.

Definition 1.1. Let M and M' be CR manifolds and let $H \colon (M, p) \to M'$, $\tilde{H} \colon (M, \tilde{p}) \to M'$ be germs of smooth CR maps at p and $\tilde{p}$, respectively. We say that H and $\tilde{H}$ are *equivalent* if there exist germs of local CR diffeomorphisms $\gamma \colon (M, p) \to (M, \tilde{p})$ and $\psi \colon (M', \tilde{H}(\tilde{p})) \to (M', H(p))$, such that

$$H = \psi \circ \tilde{H} \circ \gamma^{-1}.$$

We say that CR maps are *equivalent* if they represent two equivalent germs at some interior points in their domains of definition.

In our main theorem below, we classify the germs of CR maps from $\mathbb{S}^3$ into $\mathcal{T}$ as five equivalence classes of map germs. Their representing maps have simple formulas when being expressed in the Heisenberg model $\mathbb{H}^3 := \left\{ (z, w) \in \mathbb{C}^2 \colon \operatorname{Im} w = |z|^2 \right\}$ for $\mathbb{S}^3$ and the rational model $\mathcal{X}$ for the tube given explicitly as follows.

$$\mathcal{X} := \left\{ (z, w) \in \mathbb{C}^2 \colon v = \frac{|z|^2 + \operatorname{Re}(z^2 \bar{\zeta})}{1 - |\zeta|^2}, \ |\zeta|^2 < 1 \right\}, \quad w = u + iv.$$

An explicit local equivalence of $\mathcal{X}$ and $\mathcal{T}$ is given by Fels and Kaup in [14, Proposition 4.16].

Theorem 1.2. *Let U be an open subset of $\mathbb{H}^3$ and $H \colon U \to \mathcal{X}$ a smooth CR map. Then the following hold:*

(a) *If H is CR transversal at some point $p \in U$, then H is CR transversal on U, the germs (H, q), $q \in U$, are mutually equivalent and are equivalent to exactly one of the following four pairwise inequivalent germs at the origin:*

    (i) $\ell(z, w) = (z, 0, w)$,
    (ii) $r_1(z, w) = \left(\dfrac{z(1 + iw)}{1 - w^2}, \dfrac{2z^2}{1 - w^2}, \dfrac{w}{1 - w^2} \right)$,
    (iii) $r_{-1}(z, w) = \left(\dfrac{z(1 - iw)}{1 - w^2}, \dfrac{-2z^2}{1 - w^2}, \dfrac{w}{1 - w^2} \right)$,
    (iv) $\iota(z, w) = (2z, \ 2w, \ 2w) \big/ \left(1 + \sqrt{1 - 4w^2 - 4iz^2} \right)$.

    Consequently, H extends as an algebraic map from $\mathbb{C}^2$ to $\mathbb{C}^3$ sending $\mathbb{H}^3$ into $\mathcal{X}$.

(b) *If H is nowhere CR transversal, then for each q, the germ (H, q) is equivalent to the germ at the origin of a map $t_q \colon (z, w) \mapsto (0, \phi_q(z, w), 0)$ for a smooth CR function ϕ_q defined in a neighborhood of the origin.*

Remark 1. The smoothness assumption in Theorem 1.2 can be weakened to C^2-smoothness by results of Kossovskiy–Lamel–Xiao [30] or Lamel–Mir [32]. Regularity of CR maps into the tube over the future light cone has been studied in [37,46,16]. Moreover, by Lamel–Mir [33], formal CR transversal holomorphic maps sending $\mathbb{H}^3$ into $\mathcal{X}$ must be convergent.

As briefly mentioned above, this theorem implies a complete classification of proper holomorphic maps from $\mathbb{B}^2$ to the classical domain D_3^{IV} that extend smoothly to some boundary point. Recall [5, §V] that the Cartan's classical domain of type IV in dimension m, denoted by D_m^{IV}, is the domain of m-dimensional vectors

$$Z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$$

satisfying the inequalities

$$1 - 2Z\overline{Z}^t + \left|ZZ^t\right|^2 > 0, \quad \left|ZZ^t\right| < 1,$$

where Z^t is the $m \times 1$ transpose of Z. This domain is also called the *Lie ball* and the *complex sphere* in the literature. This domain is homogeneous: the global automorphism group $\mathrm{Aut}(D_3^{\mathrm{IV}})$ acts transitively on it. We say that two proper holomorphic maps $H, \widetilde{H} \colon \mathbb{B}^2 \to D_3^{\mathrm{IV}}$ are *equivalent* if there exist $\psi \in \mathrm{Aut}(D_3^{\mathrm{IV}})$ and $\gamma \in \mathrm{Aut}(\mathbb{B}^2)$ such that $\widetilde{H} := \psi \circ H \circ \gamma^{-1}$. Then we obtain a classification of proper holomorphic maps from $\mathbb{B}^2$ to D_3^{IV} as follows.

Corollary 1.3. *Let $H \colon \mathbb{B}^2 \to D_3^{\mathrm{IV}}$ be a proper holomorphic map which extends smoothly to some boundary point $p \in \partial \mathbb{B}^2$. Then H is equivalent to exactly one of the following four pairwise inequivalent maps:*

(i) $R_0(z, w) = \left(\dfrac{z}{\sqrt{2}}, \dfrac{2w^2 + 2w - z^2}{4(w+1)}, \dfrac{i\left(2w^2 + 2w + z^2\right)}{4(w+1)} \right),$

(ii) $P_1(z, w) = \left(zw, \dfrac{z^2 - w^2}{2}, \dfrac{i(z^2 + w^2)}{2} \right),$

(iii) $P_{-1}(z, w) = \left(z, \dfrac{w^2}{2}, \dfrac{iw^2}{2} \right),$

(iv) $I(z, w) = \left(z, w, 1 - \sqrt{1 - z^2 - w^2} \right) / \sqrt{2}.$

Remark 2. The maps R_0 and I were known earlier in the classification of isometric embeddings, studied in, e.g., [6,44], by methods that are very different from ours. They also appeared in [47] which studies the rigidity of transversal holomorphic maps sending a piece of the sphere into the smooth boundary part of a type IV domain in higher dimension. They are essentially the only isometric embeddings (up to a normalizing constant) of the respective Bergman metrics on $\mathbb{B}^2$ and D_3^{IV}. The quadratic polynomial maps $P_{\pm 1}$ are not isometric and provide counterexamples to Conjecture 2.9 in [47] of

Xiao and Yuan in the case the source has dimension $n = 2$ (the case $n = 3$ and the case of C^1-extendability are still open). In fact, there is a 1-parameter smooth family of rational proper maps R_α, $\alpha \in \mathbb{R}$, such that $R_\alpha \sim P_1$ for $\alpha > 0$ and $R_\alpha \sim P_{-1}$ for $\alpha < 0$, yet R_0 is not equivalent to R_α for $\alpha \neq 0$. In essence, the map R_0 is not "locally rigid" (see [10] for a recent discussion on the local rigidity of CR maps). This exhibits an important difference of our setting from the case $n \geqslant 4$ and the case of isometries treated in [47,44,6] in which the rigidity holds.

It is immediate to see that these four maps are pairwise inequivalent. Indeed, R_0 and I (which is irrational in all representations of the target considered in this paper) have singularities on the sphere, the image of $\mathbb{S}^3$ via P_1 lies entirely in the smooth part $\mathcal{R} \subset \partial D_3^{\mathrm{IV}}$, while P_{-1} sends the circle $\{(e^{it}, 0)\} \subset \mathbb{S}^3$ into the singular part of $\partial D_3^{\mathrm{IV}}$ and sends $\mathbb{S}^3 \setminus \{(e^{it}, 0)\}$ transversally into $\mathcal{R}$. Hence, the inequivalences are evident.

Our proof of Corollary 1.3 is independent from that of the classification of proper holomorphic maps and isometric embeddings in [6,44,47].

Remark 3. Corollary 1.3 is analogous to Faran's well-known classification of maps from $\mathbb{B}^2$ to $\mathbb{B}^3$ as a list of four equivalence classes of maps [12] (see also [42,26]). We point out, however, that there is a difference when considering *local* equivalences of map germs. For each map $H \in \{R_0, P_1, P_{-1}, I\}$ and each $p, p' \in \mathbb{S}^3$ such that $H(p), H(p') \in \mathcal{R}$, the germs (H, p) and (H, p') are equivalent. In contrast, the sphere map $(z, w) \mapsto (z, zw, w^2)$ from $\mathbb{B}^2$ to $\mathbb{B}^3$ represents inequivalent germs when the base point p varies on and off the circle of "umbilical points" $\{(0, e^{i\theta}) \in \mathbb{S}^3 : t \in \mathbb{R}\}$. The sphere map $(z, w) \mapsto (z^3, \sqrt{3}zw, w^3)$ also exhibits a similar phenomenon, see [42] and [34, Table 1].

It is interesting to point out, that unlike in the sphere case, where the only everywhere degenerate map (in the sense of [31]) is the linear map, in the setting of Theorem 1.2 all transversal CR maps are 2-nondegenerate.

Remark 4. There is a convenient way to construct proper holomorphic maps from a ball into a type IV domain of certain dimension as first noticed in [47]. In essence, it relies on an equation of the form

$$2\sum_{j=1}^{m} |h_j|^2 - \left| \sum_{j=1}^{m} h_j^2 \right|^2 = \sum_{j=1}^{m'} |f_j|^2, \tag{1.1}$$

where $H = (h_1, h_2, \ldots, h_m)$ and $F = (f_1, f_2, \ldots, f_{m'})$ are two holomorphic maps defined on some relevant open set of $\mathbb{C}^n$. If H maps $\mathbb{S}^{2n-1}$ into $\partial D_m^{\mathrm{IV}}$, then F maps $\mathbb{S}^{2n-1}$ into $\mathbb{S}^{2m'+1}$ and vice versa, if F maps $\mathbb{S}^{2n-1}$ into $\mathbb{S}^{2m'+1}$, then H maps $\mathbb{S}^{2n-1}$ into the real variety $\{\varrho_{D_m^{\mathrm{IV}}} = 0\}$ which contains $\partial D_m^{\mathrm{IV}}$. Equation (1.1) implies that $H|_{\mathbb{S}^{2n+1}}$ and $F|_{\mathbb{S}^{2n+1}}$ have the same Ahlfors tensor (defined in Section 2). If we take $H = P_1$, then (1.1) holds for $F(z, w) = (z^2, \sqrt{2}zw, w^2)$, which maps $\mathbb{S}^3$ into $\mathbb{S}^5$. Both maps have Ahlfors tensor equal to $1/2$ (when evaluated on an appropriate frame). If $H = P_{-1}$, then there is no map $F : \mathbb{S}^3 \to \mathbb{S}^5$ such that (1.1) holds, since P_{-1} has a negative Ahlfors

tensor, cf. [47, Example 2.8] and the proof of Corollary 1.3, which can never be satisfied by any sphere map, see [34].

Each of the isometric embeddings R_0 and I restricts to a CR map on an open dense subset of $\mathbb{S}^3$ having the property that a certain coefficient in its partial normal form (given below in (3.1)) vanishes. In this case we say that the map has vanishing *geometric rank* at the centered point. The notion of geometric rank in our present setting is similar to that of sphere maps, first used by Huang [20], see also [21], and has been used widely since then in the study of sphere maps. It also appears recently in [25] as a boundary characterization for isometric embeddings into indefinite hyperbolic space. We define this invariant notion for CR transversal maps into the tube over the future light cone precisely in Definition 3.2. By inspecting the special case of vanishing geometric rank in the proof of Theorem 1.2, we can observe a phenomenon that is similar to the main result in [25].

Corollary 1.4. *Let $H\colon \mathbb{B}^2 \to D_3^{\mathrm{IV}}$ be a proper holomorphic map. If H extends smoothly to a neighborhood of a point $p \in \mathbb{S}^3$ and has vanishing geometric rank at p, then H is a holomorphic isometry (with respect to the Bergman metrics). Conversely, each isometric embedding from $\mathbb{B}^2$ into D_3^{IV} extends to a real-analytic CR map from an open dense proper subset of the sphere $\mathbb{S}^3$ to the boundary of $\partial D_3^{\mathrm{IV}}$ with vanishing geometric rank at every point and is everywhere CR transversal on its domain.*

We conclude this introduction by describing our approach in the proof of Theorem 1.2 and the main difficulties we need to overcome. Similarly to, e.g. [42,47], we analyze the "mapping equation" for the components of a map H which expresses the fact that H sends $\mathbb{H}^3$ into $\mathcal{X}$. But there are differences to the positive but "low" codimension case, as treated in [47], when "rigidity" holds, i.e., when there are only "few" equivalence classes of maps: An essential ingredient for the rigidity is Huang's lemma [20] and the fact that the maps are all of vanishing "geometric rank". In our present setting (as well as in the case of sphere maps from $\mathbb{S}^3$ to $\mathbb{S}^5$, cf. [12,42]) Huang's lemma is not available and the rigidity *fails*. Thus, the solution set in the present setting is expected to be rather "large" and we need to solve the mapping equation for an *a priori* unknown number of (discrete or continuous families of) solutions.

To analyze the mapping equation, we first normalize the map (the "unknown") to a specific form, given in (3.1), using explicit formulas for the local CR automorphisms of the source and target (Lemma 2.2). In this partial normal form, the mapping equation can be viewed as either a system of infinitely many (linear and nonlinear) equations for the coefficients of the Taylor series expansions of the components the map (the jets of the map at the origin) or a system of infinitely many equations arising from differentiation and evaluation along the first Segre variety $\Sigma := \{(z,w) \in \mathbb{C}^2 \colon w = 0\}$ of $\mathbb{H}^3$ at the origin. By the 2-nondegeneracy of $\mathcal{X}$, the jets of the map of all orders are determined once the 4-jet is determined. Such a behavior is also observed in the sphere case, see [42].

A bulk of the present paper is to identify the 4-jets at the origin that arise from maps sending $\mathbb{H}^3$ into $\mathcal{X}$. This process involves analyzing the mapping equation and various derived ones. It turns out that the 4-jets of the maps can only be fully identified by analyzing all equations arising from the Taylor series expansions up to weighted order 16. There are examples of maps which satisfy all but three of weight 14 (Example 5.3). This shows that considering such high orders is in some sense necessary. Once we determine the 4-jets of the maps at the origin, and at the same time, the 2-jets along the first Segre variety, we can construct a system of three holomorphic functional equations depending on two real parameters whose unique and explicit solutions, as presented in Theorem 1.2, are genuine maps sending $\mathbb{H}^3$ into $\mathcal{X}$.

The computational complexity also poses some difficulties. To overcome this issue, we use the computer algebra system Mathematica when doing some formal differentiations, collecting terms in polynomial expressions, and computing determinants of matrices of some (relatively small) size. But we want to point out, that the operations and expressions involved in the proofs are in principle simple and elementary enough to be checked by hand.

The paper is organized as follows: In section 2 we provide the local representations of the tube over the future light cone, its automorphisms and an Ahlfors-type tensor, which allows us to distinguish equivalence classes of maps. In section 3 we give a partial normal form for the maps and introduce the geometric rank in this setting. In section 4 we give the proof of our main result Theorem 1.2 and in section 5 we discuss additional examples of maps and provide explicit biholomorphisms to show equivalence to the maps listed in Theorem 1.2.

2. Preliminaries

2.1. Models for the tube over the future light cone

Following [14] we would like to describe representations of the tube over the future light cone. A model for the three-dimensional space time is the space V of real symmetric 2×2-matrices where the time coordinate is the normalized trace of the matrices. In this model, the *future cone* is

$$\Omega := \left\{ v = \begin{pmatrix} t + x_1 & x_2 \\ x_2 & t - x_1 \end{pmatrix} : v \text{ positive definite} \right\}.$$

The smooth boundary part of Ω is the *future light cone* $\mathcal{C}$ is given explicitly by

$$t^2 = x_1^2 + x_2^2, \quad t > 0.$$

To construct the tube over $\mathcal{C}$, one identifies $V \oplus iV$ with the space of complex symmetric 2×2-matrices. The *tube over the future light cone* is given by

$$\mathcal{T} := \mathcal{C} \oplus iV = \{z \in V \oplus iV \colon \det(z + \bar{z}) = 0, \ \operatorname{Re} \operatorname{tr}(z + \bar{z}) > 0\}.$$

The tube $\mathcal{T}$ is the smooth boundary part of a tube domain with everywhere degenerate Levi-form. In fact, it is everywhere 2-nondegenerate as a CR manifold; see [11, Example 4.2.1]. It has been studied in many papers [11,27,14]. Our interest in this tube is that it is *holomorphically homogeneous*, i.e., its CR automorphism group acts transitively on it, and has quite "large" stability group at each point. Moreover, from Fels–Kaup [14], $\mathcal{T}$ has a local rational model similar as in the case of the sphere. Namely, at every point it is locally CR equivalent to the real hypersurface $\mathcal{X}$ near the origin in $\mathbb{C}_{z,\zeta,w}$ given by

$$v = \frac{|z|^2 + \operatorname{Re}(z^2\bar{\zeta})}{1 - |\zeta|^2}, \quad w = u + iv.$$

In fact, the rational map

$$\Phi : (z, \zeta, w) \mapsto \frac{1}{1 + \zeta} \begin{pmatrix} z^2 - i\zeta w - iw & \sqrt{2}w \\ \sqrt{2}w & 1 - \zeta \end{pmatrix}$$

sends $\mathcal{X}$ into $\mathcal{T}$ [14, Proposition 4.16] and is locally biholomorphic on a dense set of $\mathbb{C}^3$. In this model, the automorphism group and the stability group have quite simple representations. In this paper, we shall work extensively with this model.

For the tube model, the maps are given as follows.

Theorem 2.1. *Let $U \subset \mathbb{H}^3$ be a connected open set and $H \colon U \to \mathcal{T}$ be a smooth CR map. Then*

(a) *If H is CR transversal at some point $p \in U$, then for all $q \in U$, the germs (H, q) are mutually equivalent and is equivalent to one of the following germs at the origin:*

(i) $\mathcal{R}_0(z, w) = \begin{pmatrix} z^2 - iw & \sqrt{2}z \\ \sqrt{2}z & 1 \end{pmatrix}$,

(ii) $\mathcal{R}_1(z, w) = \dfrac{1}{1 - w^2 + 2z^2} \begin{pmatrix} z^2 - iw & \sqrt{2}z(1 + iw) \\ \sqrt{2}z(1 + iw) & 1 - w^2 - 2z^2 \end{pmatrix}$,

(iii) $\mathcal{R}_{-1}(z, w) = \dfrac{1}{1 - w^2 - 2z^2} \begin{pmatrix} z^2 - iw & \sqrt{2}z(1 - iw) \\ \sqrt{2}z(1 - iw) & 1 - w^2 + 2z^2 \end{pmatrix}$,

(iv) $\mathcal{I}(z, w) = \dfrac{1}{1 + 2w + \varepsilon} \begin{pmatrix} -i(1 + 2w - \varepsilon) & 2\sqrt{2}z \\ 2\sqrt{2}z & 1 - 2w + \varepsilon \end{pmatrix}$, $\varepsilon := \sqrt{1 - 4w^2 - 4iz^2}$.

(b) *If H is nowhere CR transversal, then for each $p \in U$, (H, p) is equivalent to a germ at the origin of the form $\phi_p(z, w) = \begin{pmatrix} 0 & 0 \\ 0 & \varphi_p \end{pmatrix}$ for some smooth CR function φ_p.*

It is well-known that $\mathcal{T}$ is locally CR equivalent to the smooth boundary part of Cartan's classical domain of type IV. Recall from [5, Section V] that the symmetric

domain of type IV of m dimension, denoted by D_m^{IV}, is the domain of m-dimensional vectors

$$Z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$$

satisfying the conditions

$$1 - 2Z\overline{Z}^t + \left|ZZ^t\right|^2 > 0, \quad \left|ZZ^t\right| < 1,$$

where Z^t is the $m{\times}1$ transpose of Z. This domain is also called the *Lie ball* or the *complex sphere* in the literature. The complex 3-dimensional case (i.e. $m = 3$) is rather special as D_3^{IV} is also equivalent to the classical domain of type II, denoted by D_2^{II}, consisting of symmetric 2×2-matrices Z such that $I_{2\times 2} - Z\bar{Z}$ is positive definite. Explicitly,

$$\varrho_{D_3^{\mathrm{IV}}} = \det\left(I_{2\times 2} - \begin{pmatrix} z_1 + iz_3 & iz_2 \\ iz_2 & z_1 - iz_3 \end{pmatrix} \cdot \begin{pmatrix} \bar{z}_1 - i\bar{z}_3 & -i\bar{z}_2 \\ -i\bar{z}_2 & \bar{z}_1 + i\bar{z}_3 \end{pmatrix} \right)$$

while the trace of the matrix in the parenthesis is just $1 - |ZZ^t|^2$. The biholomorphisms of D_3^{IV} and D_2^{II} are well-known, see [19].

2.2. Automorphisms of the tube over the future light cone

The local CR automorphisms of the tube over the future light cone are well-studied. They are the restrictions of birational but not necessarily biholomorphic transformations of $\mathbb{C}^3$; a local CR equivalence of $\mathcal{T}$ does not necessarily extend to a global CR equivalence [27]. By homogeneity, the local CR equivalences of $\mathcal{T}$ are determined once the automorphism group $\mathrm{Aut}(\mathcal{T}, p)$ of the germs at an arbitrary point p is determined. In fact, $\mathrm{Aut}(\mathcal{T}, p)$ is a solvable real Lie group of dimension 5 (isomorphic to the stability group of a point on a sphere in $\mathbb{C}^2$ [4, Theorem 5]). For our purpose, we shall describe this group explicitly in the local rational model of Fels–Kaup [14]. The result of Fels–Kaup is presented in [29] by writing the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

The stability group G_0^c generated by the component

$$\mathfrak{g}_0^c := \mathrm{span}\left\{ z\frac{\partial}{\partial z} + 2w\frac{\partial}{\partial w}, \ iz\frac{\partial}{\partial z} + 2i\zeta\frac{\partial}{\partial \zeta} \right\}$$

is parametrized by $\lambda > 0$ and $\varphi \in \mathbb{R}$ as follows

$$(z, \zeta, w) \mapsto \left(\lambda e^{i\varphi} z, e^{2i\varphi}\zeta, \lambda^2 w \right).$$

The component $\mathfrak{g}_2$ is

$$\mathfrak{g}_2 = \operatorname{span}\left\{ zw\frac{\partial}{\partial z} - iz^2\frac{\partial}{\partial \zeta} + w^2\frac{\partial}{\partial w} \right\}.$$

Integrating yields a 1-parameter subgroup and we get

$$(z, \zeta, w) \mapsto \big(z, \zeta - sw\zeta - isz^2, w\big)/(1 - sw), \quad s \in \mathbb{R}.$$

The component $\mathfrak{g}_1$ is given by

$$\mathfrak{g}_1 = \operatorname{span}\bigg\{ (z^2 + iw(\zeta + 1))\frac{\partial}{\partial z} + 2z(\zeta + 1)\frac{\partial}{\partial \zeta} + 2zw\frac{\partial}{\partial w},$$

$$(iz^2 - w(\zeta - 1))\frac{\partial}{\partial z} + 2iz(\zeta - 1)\frac{\partial}{\partial \zeta} + 2izw\frac{\partial}{\partial w} \bigg\}.$$

Integrating the first vector field, we obtain

$$(z, \zeta, w) \mapsto \big(z(1 - tz) + it(1 + \zeta)w, -\delta_1 + \zeta + 1, w\big)/\delta_1, \ \delta_1 = (1 - tz)^2 - i(\zeta + 1)t^2w.$$

Integrating the second vector field, we get

$$(z, \zeta, w) \mapsto \big(z(1 - irz) - r(\zeta - 1)w, \delta_2 + \zeta - 1, w\big)/\delta_2, \quad \delta_2 = (1 - irz)^2 + i(\zeta - 1)r^2w.$$

Composing the automorphisms obtained above, we have

Lemma 2.2 *(Stability group). Let $a \in \mathbb{C}, t \in \mathbb{R}, u \in \mathbb{C}, |u| = 1, \lambda > 0$ and*

$$\delta = 1 - 2i\bar{a}z - (t + i|a|^2)w + i\bar{a}^2(w\zeta + iz^2).$$

Then the stability group $\operatorname{Aut}(\mathcal{X}, 0)$ is the 5 dimensional group consisting of the automorphisms

$$\psi_{a,t,u,\lambda}\colon \begin{cases} \tilde{z} = \lambda u \left(z + aw - \bar{a}(w\zeta + iz^2) \right)\delta^{-1} \\ \tilde{w} = \lambda^2 w\delta^{-1} \\ \tilde{\zeta} = u^2 \left(\zeta - 2iaz - ia^2w - (t - i|a|^2)(w\zeta + iz^2) \right)\delta^{-1} \end{cases} \tag{2.1}$$

Infinitesimal automorphisms, which do not vanish at the origin, give rise to automorphisms moving points. We provide them in the following:

The vector field $\partial/\partial w$ spanning $\mathfrak{g}_{-2}$ integrates to the 1-parameter family

$$(z, \zeta, w) \mapsto (z, \zeta, w + s) \quad \text{for} \quad s \in \mathbb{R}. \tag{2.2}$$

The component $\mathfrak{g}_{-1}$ is given by

$$\mathfrak{g}_{-1} = \operatorname{span}\left\{ (1 - \zeta)\frac{\partial}{\partial z} + 2iz\frac{\partial}{\partial w}, i(1 + \zeta)\frac{\partial}{\partial z} + 2z\frac{\partial}{\partial w} \right\},$$

whose vector fields integrate to

$$(z, \zeta, w) \mapsto (z + t'(1 - \zeta), \zeta, w + 2it'z + it'^2(1 - \zeta)), \qquad t' \in \mathbb{R},$$
$$(z, \zeta, w) \mapsto (z + is'(1 + \zeta), \zeta, w + 2s'z + is'^2(1 + \zeta)), \qquad s' \in \mathbb{R}.$$

Combining the three above automorphisms gives the following 3-parameter family which generates the Lie group corresponding to $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$,

$$\tau_{b,r} \colon (z, \zeta, w) \mapsto (z + b - \bar{b}\zeta, \; \zeta, \; w + r + i|b|^2 + 2i\bar{b}z + i\bar{b}^2/2 - ib^2/2 - i\bar{b}^2\zeta), \quad (2.3)$$

for $b \in \mathbb{C}$ and $r \in \mathbb{R}$, which maps 0 to $(b, 0, r + i|b|^2 + i\bar{b}^2/2 - ib^2/2)$.

The component $\mathfrak{g}_0^s$ is given by

$$\mathfrak{g}_0^s = \mathrm{span}\left\{ -z\zeta\frac{\partial}{\partial z} + (1 - \zeta^2)\frac{\partial}{\partial \zeta} + iz^2\frac{\partial}{\partial w}, \; iz\zeta\frac{\partial}{\partial z} + i(1 + \zeta^2)\frac{\partial}{\partial \zeta} + z^2\frac{\partial}{\partial w} \right\}.$$

If we write

$$\varepsilon_1 = 1 + \lambda' + (\lambda' - 1)\zeta, \qquad \lambda' > 0,$$

then the first vector field integrates to

$$(z, \zeta, w) \mapsto \left(2\sqrt{\lambda'}z, \varepsilon_1' + 2(\zeta - 1), (\lambda' + 1)w + (\lambda' - 1)(w\zeta + iz^2) \right)/\varepsilon_1. \qquad (2.4)$$

If we write

$$\varepsilon_2 = 1 + \mu' - i(\mu' - 1)\zeta, \qquad \mu' > 0,$$

then the second vector field integrates to

$$(z, \zeta, w) \mapsto \left(2\sqrt{\mu'}z, i(\mu' - 1 - i(\mu' + 1)\zeta), (\mu' + 1)w - i(\mu' - 1)(w\zeta + iz^2) \right)/\varepsilon_2.$$

If we compose the two 1-parameter families of automorphisms and write,

$$c' = 1 + i(\mu' + \lambda') + \mu'\lambda', \qquad d' = 1 + i(\mu' - \lambda') - \mu'\lambda', \qquad \gamma' = c' - d'\zeta,$$

we obtain the following 2-parameter family of automorphisms

$$\psi \colon (z, \zeta, w) \mapsto \left(2\sqrt{\mu'\lambda'}(1 + i)z, i(\bar{c}'\zeta - \bar{d}'), c'w - d'(w\zeta + iz^2) \right)/\gamma',$$

which sends 0 to $(0, -i\bar{d}'/c', 0)$.

As an application of these explicit representations of the transitive automorphisms of $\mathcal{X}$, we explicitly show in Example 5.2 below, that the rational map $R : \mathbb{B}^2 \to D_3^{\mathrm{IV}}$

obtained in [47, Theorem 1.4] corresponds to the linear map ℓ, listed in Theorem 1.2, as a map from $\mathbb{H}^3$ into $\mathcal{X}$.

To deduce Corollary 1.3 from Theorem 1.2, we shall use the local equivalence of the rational model $\mathcal{X}$ and the smooth boundary part $\mathcal{R}$ of D_3^{IV} together with a well-known fact that (in contrast with the local CR equivalences of the rational model) each local CR equivalence of $\mathcal{R}$ extends to a birational transformation of $\mathbb{C}^3$ that restricts to automorphisms of D_3^{IV} [14, Section 6], [27]. In the case of the sphere and Heisenberg hypersurface, this extension phenomenon has been known since the works of Poincaré, Tanaka, and Alexander, cf. [41,2]. For general classical domains of rank $\geqslant 2$, the same result was established by Mok–Ng [40].

Theorem 2.3. *Let $p \in \mathcal{R}$ be a point in the smooth boundary part $\mathcal{R}$ of D_3^{IV} and let U be a neighborhood of p in $\mathcal{R}$. If $H \colon U \to \mathcal{R}$ is a local CR diffeomorphism, then there exists a rational map $\widetilde{H}$ such that $\widetilde{H}|_{D_3^{\mathrm{IV}}}$ is an isometry of D_3^{IV} with respect to the Bergman metric and $H = \widetilde{H}|_U$.*

We should also mention a well-known theorem of Tumanov–Khenkin [43] which gives a similar conclusion when the smooth boundary part is replaced by the Shilov boundary, the "skeleton" in the terminology of [43], of the domain. For the sake of completeness, we give an elementary proof that is based on Lemma 2.2 for the special case of D_3^{IV}.

Proof. Since the automorphisms of D_3^{IV} act transitively on $\mathcal{R}$, we may assume that $H(p) = p = (0, \frac{1}{2}, \frac{i}{2})$. The rational map Φ

$$\Phi(z, \zeta, w) = \left(\frac{2iz}{2i + w}, \ \frac{2i - w - 2i\zeta - (w\zeta + iz^2)}{2(2i + w)}, \ \frac{i\left(2i - w + 2i\zeta + (w\zeta + iz^2)\right)}{2(2i + w)} \right) \tag{2.5}$$

sends a neighborhood V of the origin biholomorphically onto some neighborhood U of p with $\Phi(0,0,0) = p$ and locally $\mathcal{X}$ into $\mathcal{R}$. Then $\gamma := \Phi^{-1} \circ H \circ \Phi$ defines a local CR diffeomorphism of $\mathcal{X}$ fixing the origin, i.e., γ represents an element of the stability group $\mathrm{Aut}(\mathcal{X}, 0)$ and has the form (2.1) for an appropriate quadruple of parameters a, u, t, and λ. Therefore, $H = \Phi \circ \gamma \circ \Phi^{-1}$ agrees with a rational automorphism (in fact, an isometry) of D_3^{IV} as can be checked directly. Indeed, explicitly one can take

$$\Phi^{-1}(z_1, z_2, z_3) = \left(\frac{2z_1}{1 + z_2 - iz_3}, \ -\frac{z_2 + iz_3 + z_1^2 + z_2^2 + z_3^2}{1 + z_2 - iz_3}, \ \frac{2i(1 - z_2 + iz_3)}{1 + z_2 - iz_3} \right) \tag{2.6}$$

Then $\Phi \circ \gamma \circ \Phi^{-1}$ is a second degree rational mapping of the form

$$\widetilde{H} = \Phi \circ \gamma \circ \Phi^{-1} = \left(\frac{\psi_1}{\Delta}, \ \frac{\psi_2}{2\Delta}, \ \frac{\psi_3}{2\Delta} \right) \tag{2.7}$$

where the components of the map are

$$\psi_1 = 2u\lambda(ia + z_1 + 2z_2 \operatorname{Im}(a) - z_3 \operatorname{Re}(a) - i\bar{a}(z_1^2 + z_2^2 + z_3^2)),$$

$$\psi_2 = 1 + 2|a|^2 - \lambda^2 - 2it - 2a^2u^2 + 4iz_1\left(au^2 - \bar{a}\right)$$

$$+ z_2\left(u^2\left(2a^2 - 2|a|^2 + \lambda^2 - 2it + 1\right) + 1 - 2|a|^2 + \lambda^2 + 2\bar{a}^2 + 2it\right)$$

$$- iz_3\left(u^2\left(2a^2 + 2|a|^2 - \lambda^2 + 2it - 1\right) + 1 - 2|a|^2 + \lambda^2 - 2\bar{a}^2 + 2it\right)$$

$$+ \left(u^2\left(1 + 2|a|^2 - \lambda^2 + 2it\right) - 2\bar{a}^2\right)\left(z_1^2 + z_2^2 + z_3^2\right),$$

$$\psi_3 = i\big(1 + 2|a|^2 - \lambda^2 - 2it + 2a^2u^2 - 4iz_1(au^2 + \bar{a})$$

$$+ z_2(1 - 2|a|^2 + \lambda^2 + 2\bar{a}^2 + 2it - u^2(1 - 2|a|^2 + \lambda^2 + 2a^2 - 2it))$$

$$- iz_3\left(u^2\left(1 - 2|a|^2 + \lambda^2 - 2a^2 - 2it\right) + 1 - 2|a|^2 + \lambda^2 - 2\bar{a}^2 + 2it\right)$$

$$- \left(u^2\left(1 + 2|a|^2 - \lambda^2 + 2it + 1\right) + 2\bar{a}^2\right)\left(z_1^2 + z_2^2 + z_3^2\right)\big),$$

and the common denominator is

$$\Delta = (1 + 2|a|^2 + \lambda^2 - 2it) - 2\bar{a}^2(z_1^2 + z_2^2 + z_3^2) - 4i\bar{a}z_1$$

$$+(1 - 2|a|^2 - \lambda^2 + 2\bar{a}^2 + 2it)z_2 - i(1 - 2|a|^2 - \lambda^2 - 2\bar{a}^2 + 2it)z_3.$$

Equation (2.7) gives the most general parametric formula for the CR automorphisms of the germ $(\mathcal{R}, p)$.

To show that the rational map $\widetilde{H}$ restricts to an automorphism of D_3^{IV}, it suffices to show that Δ does not vanish on D_3^{IV}. If we write

$$W = \left(\frac{-2ia}{1 + 2|a|^2 + \lambda^2 + 2it}, \ \frac{\lambda^2 - 1 + 2|a|^2 - 2a^2 + 2it}{2(1 + 2|a|^2 + \lambda^2 + 2it)}, \ \frac{i(\lambda^2 - 1 + 2|a|^2 + 2a^2 + 2it)}{2(1 + 2|a|^2 + \lambda^2 + 2it)}\right)$$

then $W \in D_3^{\mathrm{IV}}$, $\widetilde{H}(W) = (0, 0, 0)$ and furthermore

$$\Delta = (1 + 2|a|^2 + \lambda^2 - 2it)\left(1 - 2\overline{W}^t Z + \overline{(W^t W)}\left(Z^t Z\right)\right).$$

But from the explicit formula for the Bergman kernel $K_{D_3^{\mathrm{IV}}}(Z, W)$ of D_3^{IV} (see [19]), we have

$$K_{D_3^{\mathrm{IV}}}(Z, W) = \mathrm{volume}(D_3^{\mathrm{IV}})^{-1}\left(\frac{\Delta}{1 + 2|a|^2 + \lambda^2 - 2it}\right)^{-3}$$

and the non-vanishing of Δ on D_3^{IV} is equivalent to the well-known fact that D_3^{IV}, being a homogeneous complete circular domain, has non-vanishing Bergman kernel (i.e., is a Lu Qi-Keng domain). □

2.3. An invariant for CR maps into the tube over the future light cone

In this section, we analyze a tensor attached to each transversal CR map into the tube over the future light cone or into $\mathcal{R}$ that is similar to the CR Ahlfors derivative of CR immersions studied recently by Lamel and the second author. We refer the reader to [34] for some details about the origin, motivation, and a general differential geometric construction of the CR Ahlfors tensor for CR immersions in the strictly pseudoconvex setting. In our current setting, the construction is rather ad-hoc, yet the resulting tensor is still useful for our purpose. Namely, it has an invariant property and gives an easy way to distinguish inequivalent maps into the tube over the future light cone. Indeed, we shall use it in the proof of Corollary 1.3.

Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface defined by $\varrho_M = 0$ and $H \colon M \to \mathcal{R}$ a CR transversal CR map extending holomorphically to a neighborhood of M. By the CR transversality, there is a real-valued smooth function v such that

$$\varrho_{D_m^{\mathrm{IV}}} \circ H = \pm e^v \varrho_M, \quad \varrho_{D_m^{\mathrm{IV}}} := 1 - 2Z\overline{Z}^t + |ZZ^t|^2.$$

Without loss of generality, we assume that the positive sign occurs. If M is a sphere we take $\varrho_M := 1 - |z|^2$ and fix the defining function of the target $\mathcal{R}$ to be $\varrho_{D_m^{\mathrm{IV}}}$. Then we define the *Ahlfors tensor* $\mathcal{A}(H)$ by

$$\mathcal{A}(H) = v_{Z\overline{Z}}\big|_{T^{(1,0)}M \times T^{(0,1)}M}, \tag{2.8}$$

where $v_{Z\overline{Z}}$ is the complex Hessian of v. For any $\psi \in \mathrm{Aut}(D_m^{\mathrm{IV}})$, we have

$$\varrho_{D_m^{\mathrm{IV}}} \circ \psi = |q|^2 \varrho_{D_m^{\mathrm{IV}}}$$

for some rational function q having no pole and zero on $\mathcal{R}$. Thus, arguing as in [34], we can see that $\mathcal{A}(H)$ is invariant with respect to composing with automorphisms of D_m^{IV}, i.e., on the smooth boundary part $\mathcal{R}$,

$$\mathcal{A}(H) = \mathcal{A}(\psi \circ H), \quad \forall\, \psi \in \mathrm{Aut}(D_m^{\mathrm{IV}}). \tag{2.9}$$

In fact, if $\tilde{H} := \psi \circ H$ and $\varrho_{D_m^{\mathrm{IV}}} \circ \tilde{H} = e^{\tilde{v}} \varrho_M$, then

$$\varrho_{D_m^{\mathrm{IV}}} \circ \tilde{H} = \left(\varrho_{D_m^{\mathrm{IV}}} \circ \psi\right) \circ H = \left(|q|^2 \varrho_{D_m^{\mathrm{IV}}}\right) \circ H = |q \circ H|^2 \left(\varrho_{D_m^{\mathrm{IV}}} \circ H\right) = |q \circ H|^2 e^v \varrho_M,$$

and, consequently,

$$\tilde{v} = v + \log |q \circ H|^2.$$

Since q has no pole nor zero on $\mathcal{R}$, $\log|q \circ H|^2$, being the logarithm of the modulus of a non-vanishing holomorphic function, has vanishing complex Hessian on a neighborhood of the sphere and hence (2.9) follows.

Similarly, on the sphere,

$$\mathcal{A}(H \circ \gamma^{-1}) = |\tilde{q}|^2 \mathcal{A}(H)$$

holds for every automorphism γ of the ball with a non-vanishing CR function $\tilde{q}$. Consequently, the vanishing of the Ahlfors tensor is invariant with respect to composing and pre-composing with automorphisms of the source and target. These invariant properties do not hold for general defining functions of the source and target.

Let $\{Z_\alpha\}$ be a local frame for $T^{(1,0)}(M)$, then the Ahlfors tensor is represented by a Hermitian $n \times n$-matrix whose rank is called the *Ahlfors rank* of H at p and denoted by $\mathrm{Rk}_{\mathcal{A}}(H)$. The Ahlfors rank is invariant with respect to composition with the source and target automorphisms. Precisely,

$$\mathrm{Rk}_{\mathcal{A}}(H)|_p = \mathrm{Rk}_{\mathcal{A}}(\psi \circ H \circ \gamma^{-1})\big|_{\gamma(p)}.$$

If M is the Heisenberg hypersurface and $H \colon \mathbb{H}^3 \to \mathcal{X}$ is a smooth CR map, then we can define the Ahlfors tensor of H which has the desired invariant property by using the defining function $\varrho_{\mathbb{H}^3} = \mathrm{Im}\, w - |z|^2$ for the source and the defining function

$$\widetilde{\varrho}_{\mathcal{X}}(z, \zeta, w) = \frac{(1 - |\zeta|^2)\,\mathrm{Im}\, w - |z|^2 - \mathrm{Re}(\zeta \bar{z}^2)}{|w + 2i|^2}$$

for the target. In fact, with this choice of the defining function for $\mathcal{X}$, we have from the explicit formulas for the local CR automorphisms in section 2.2 that

$$\widetilde{\varrho}_{\mathcal{X}} \circ \psi = |q|^2 \widetilde{\varrho}_{\mathcal{X}}$$

for some holomorphic (in fact, rational) function q. The corresponding Ahlfors tensor and its rank are invariant with respect to the action of the CR automorphisms groups of both source and target.

3. A partial normalization and the notion of geometric rank

Let $H \colon \mathbb{H}^3 \to \mathcal{X}$ be a smooth CR map from the Heisenberg hypersurface into the rational model $\mathcal{X}$ of the tube over the future light cone. We use the automorphisms of the stability groups and normalize H so that its 2-jet has a specific form.

Remark 5. Throughout this and the preceding section we use the following notation: Let h be a germ at $(0,0)$ of a holomorphic function in $\mathbb{C}^2$, depending on variables (z, w) in $\mathbb{C}^2$. Instead of writing $\frac{\partial^{k+\ell}h}{\partial z^k \partial w^\ell}$ for $k, \ell \in \mathbb{N}$, we write $h_{z\cdots z w \cdots w}$ and its evaluation at $(0,0)$ we denote by $h^{(k,\ell)}$.

Theorem 3.1. *Let $p \in \mathbb{H}^3$ and $H = (f, \phi, g)$ be a germ at p of a smooth CR map sending $\mathbb{H}^3$ into $\mathcal{X}$. Assume that H is CR transversal at p. Then the germ (H, p) is equivalent to the germ at the origin of a CR map $\widetilde{H} = (\widetilde{f}, \widetilde{\phi}, \widetilde{g})$ satisfying the following properties:*

$$\widetilde{f}(z, w) = z + \frac{i}{2}\alpha zw + \nu w^2 + O(3),$$
$$\widetilde{\phi}(z, w) = \lambda w + \alpha z^2 + \mu zw + \sigma w^2 + O(3), \qquad (3.1)$$
$$\widetilde{g}(z, w) = w + O(3),$$

where $\alpha \in \mathbb{R}$, $\lambda, \nu, \mu, \sigma \in \mathbb{C}$, and $O(3)$ denotes certain power series centered at the origin whose terms are of total degree greater than or equal to 3. Furthermore, we may assume $\lambda \in \{0, 1\}$, and if $\lambda = \mu = 0$, we may assume $\alpha \in \{-1, 0, 1\}$.

Proof. By the transitivity of the automorphisms of $\mathbb{H}^3$ and $\mathcal{X}$ we can assume $H(0) = 0$. It is easy to conclude directly from the mapping equation, that $g(z, 0) = 0$ and $g^{(0,1)} = |f^{(1,0)}|^2$. By the transversality of H we have $f^{(1,0)} \neq 0$ and $g^{(0,1)} > 0$. We write $H_{k+1} := \varphi'_k \circ H_k \circ \varphi_k^{-1}$, where φ_k^{-1} and φ'_k belong to $\mathrm{Aut}(\mathbb{H}^3, 0)$ and $\mathrm{Aut}(\mathcal{X}, 0)$ respectively, $k \in \mathbb{N}$ and $H_0 := H$. The automorphism φ'_k has the form of (2.1) with suitable parameters. That is

$$\varphi'_k = \psi_{a', t', u', s'},$$

where ψ is given in (2.1). Similarly, we use the explicit formulas for the stability group $\mathrm{Aut}(\mathbb{H}^3, 0)$ of $\mathbb{H}^3$ and write

$$\gamma_{a,t,u,s}(z, w) := \frac{(su(z + aw), s^2 w)}{1 - 2i\bar{a}z + (t - i|a|^2)w},$$

for an element in $\mathrm{Aut}(\mathbb{H}^3, 0)$, where $a \in \mathbb{C}, t \in \mathbb{R}, |u| = 1$ and $s > 0$. We set $\varphi_k := \gamma_{a,t,u,s}$. We have

$$H_{1z}(0) = \left(uu' ss' f^{(1,0)}, uu'^2 s(-2ia' f^{(1,0)} + \phi^{(1,0)}), 0\right).$$

Since $f^{(1,0)} \neq 0$, by choosing

$$u = \frac{\overline{f^{(1,0)}}}{u'|f^{(1,0)}|}, \quad s = \frac{1}{s'|f^{(1,0)}|}, \quad a' = -\frac{i\phi^{(1,0)}}{2f^{(1,0)}},$$

we can assume that $f_1^{(1,0)} = 1$ and $\phi_1^{(1,0)} = 0$. This implies $g_1^{(0,1)} = 1$. Considering $H_2 = \varphi'_1 \circ H_1 \circ \varphi_1^{-1}$ with $u = 1/u', s = 1/s'$ and $a' = 0$, we compute

$$f_2^{(0,1)} = a + \frac{u' f_1^{(0,1)}}{s'},$$

and set

$$a = -\frac{u' f_1^{(0,1)}}{s'},$$

to obtain $f_2^{(0,1)} = 0$. Differentiating the mapping equation for H_2 twice with respect to $\bar{w}$ gives $\operatorname{Im} g_2^{(0,2)} = 0$. Differentiating the mapping equation with respect to z and $\bar{w}$ gives $g_2^{(1,1)} = 0$ and with respect to z twice and $\bar{z}$ yields $f_2^{(2,0)} = 0$. If we take $H_3 = \varphi_2' \circ H_2 \circ \varphi_2^{-1}$ with $u = 1/u', s = 1/s'$ and $a = a' = 0$, we obtain

$$g_3^{(0,2)} = -2t + (2t' + g_2^{(0,2)})/s'^{\,2},$$

which, after setting

$$t = \frac{2t' + g_2^{(0,2)}}{2s'^{\,2}},$$

implies that $g_3^{(0,2)} = 0$. If we plug H_3 into the mapping equation and differentiate with respect to $z, \bar{z}$ and $\bar{w}$ we obtain that $\operatorname{Re} f_3^{(1,1)} = 0$. Moreover, if we differentiate the mapping equation twice with respect to z and $\bar{z}$ we get $\operatorname{Re} \phi_3^{(2,0)} = 2 \operatorname{Im} f_3^{(1,1)}$. In the last step, we consider $H_4 = \varphi_3' \circ H_3 \circ \varphi_3^{-1}$ with $u = 1/u', s = 1/s', a = a' = 0$ and $t = t'/s'$ to obtain

$$\phi_4^{(2,0)} = \frac{-2it' + \phi_3^{(2,0)}}{s'^{\,2}},$$

which, after setting

$$t' = \frac{\operatorname{Im} \phi_3^{(2,0)}}{2},$$

implies that $\operatorname{Im} \phi_4^{(2,0)} = 0$. Finally, we compute

$$\phi_4^{(0,1)} = \frac{u'^{\,2} \phi_3^{(0,1)}}{s'^{\,2}}, \qquad \phi_4^{(2,0)} = \frac{4 \operatorname{Im} f_3^{(1,1)}}{s'^{\,2}},$$

which implies the remaining normalization conditions, when setting $\alpha = \operatorname{Re} \phi_4^{(2,0)}/2$ and choosing s' and u' accordingly. $\quad\square$

In the partial normal form (3.1), the vanishing of the coefficient α is an invariant property of the map germ. Thus, in analogy with the case of sphere maps [20,21], we make the following definition.

Definition 3.2. Let $H \colon U \subset \mathbb{H}^3 \to \mathcal{X}$ be a smooth transversal CR map and $p \in U$. We say that H has *geometric rank* zero at p if H can be brought into the partial normal form (3.1) with $\alpha = 0$. Otherwise we say that H has *geometric rank* 1 at p.

The geometric rank is precisely the Ahlfors rank defined in Section 2.3 but the latter can be computed easily (without the use of explicit formulas for the CR automorphisms). This relation was first noticed by Lamel and the second author in the case of sphere maps (cf. [34]). Thus the invariant property of the geometric rank also follows from that of the Ahlfors tensor. Indeed, if H is given as in (3.1) and if

$$\widetilde{\varrho}_{\mathcal{X}} \circ H = \widetilde{Q} \cdot \varrho_{\mathbb{H}^3},$$

then at the origin,

$$\widetilde{Q} = 2, \quad \widetilde{Q}_z = 0, \quad \widetilde{Q}_{z\bar{z}} = 2\alpha.$$

Hence, for the defining functions as above and for $L := \partial_z - 2i\bar{z}\partial_w$ a section of $T^{(1,0)}(\mathbb{H}^3)$, we have

$$\mathcal{A}(H)(L,\overline{L})\big|_0 = \left(\log \widetilde{Q}\right)_{Z\overline{Z}}(L,\overline{L})\bigg|_0 = \widetilde{Q}^{-2}\widetilde{Q}_{z\bar{z}}\big|_0 = \frac{\alpha}{2}.$$

The relation also holds at an arbitrary point $p \in \mathbb{H}^3$ by the invariant property of the left-hand side.

4. Proofs of Theorem 1.2 and its corollaries

A bulk of this section is devoted to the proof of Theorem 1.2. Our strategy can be summarized as follows. Let Σ be the first Segre set of $\mathbb{H}^3$ at the origin, i.e.,

$$\Sigma = \{(z,0) \in \mathbb{C}^2\}.$$

We then explicitly compute the map H along Σ, that is $H(z,0)$, by analyzing the mapping equation (4.2). From these explicit formulas for $H(z,0)$, we establish a holomorphic functional equation for the components as in Lemma 4.9. We also compute the derivatives H_w and H_{ww} along Σ (see Lemma 4.4 for one instance) from which we obtain two other functional equations (as in Lemma 4.19) depending on several parameters. Additionally we need to distinguish two main cases in section 4.1 and section 4.2, leading to irrational and rational maps respectively. In the first case the system of these three equations determines the irrational map uniquely. In the second case we need to determine all sets of parameters (starting with Lemma 4.20) that are involved in the system so that, for each set, the unique solution of the system is a genuine map from $\mathbb{H}^3$ into $\mathcal{X}$. The second case itself leads to several subcases, which give us all rational maps.

Lemma 4.1. *Let $H = (f, \phi, g)$ be a map given in the normal form (3.1). Let $P = P_H$ be the rational map depending on the 2-jet of H given by*

$$P(z; X) = P_{\alpha,\lambda,\sigma,\mu,\nu}(z; X) = \left(X, \frac{2z(4\bar{\nu}z + \alpha)X + \left(4\bar{\sigma}z^2 + 2i\bar{\mu}z - \alpha\right)X^2}{1 - 4i\bar{\lambda}z^2}, 0 \right).$$

Then

$$H\big|_{\Sigma} = P\left(z; \frac{2z}{1 + \sqrt{1 - 4i\bar{\lambda}z^2}} \right). \tag{4.1}$$

Thus, H is uniquely determined along Σ by its 2-jet at the origin.

Proof. Consider the following defining function of the rational model $\mathcal{X}$ of the tube over the future light cone:

$$\widetilde{\varrho}(z, \zeta, w) = -\frac{i}{2}(w - \overline{w})(1 - |\zeta|^2) - |z|^2 - \operatorname{Re}\left(z^2\overline{\zeta}\right).$$

If H sends $\mathbb{H}^3$ into $\mathcal{X}$, then there is a function $Q(z, w, \bar{z}, \bar{w})$ near 0 such that $\widetilde{\varrho}(H) = Q\varrho$. Explicitly

$$(g - \bar{g})(1 - |\phi|^2) - 2i|f|^2 - 2i\operatorname{Re}\left(f^2\bar{\phi}\right) = Q(z, w, \bar{z}, \bar{w})(w - \bar{w} - 2i|z|^2). \tag{4.2}$$

If H is real-analytic at 0, then H extends holomorphically to a neighborhood of the origin and Q is real-analytic on some neighborhood of the origin. If H is smooth, then we can view H and Q as formal power series in $z, w, \bar{z}, \bar{w}$. In both cases, we can treat z and $\bar{z}$ as separate variables and (4.2) holds as an identity of formal power series.

Setting $\bar{z} = \bar{w} = 0$ in (4.2), we have

$$g(z, w) = wQ(z, w, 0, 0).$$

We introduce the following auxiliary holomorphic functions (or formal power series)

$$r(z, w) = \frac{\partial Q}{\partial \bar{w}}(z, w, 0, 0), \quad s(z, w) = \frac{i}{2}\frac{\partial Q}{\partial \bar{z}}(z, w, 0, 0). \tag{4.3}$$

Differentiating the mapping equation in $\bar{w}$ we obtain

$$-(1 - |\phi|^2)\bar{g}_{\bar{w}} - \phi\bar{\phi}_{\bar{w}}(g - \bar{g}) - 2if\bar{f}_{\bar{w}} - 2i\bar{f}\bar{f}_{\bar{w}}\phi - if^2\bar{\phi}_{\bar{w}} = Q_{\bar{w}}\varrho - Q.$$

Evaluating at $\bar{z} = \bar{w} = 0$, using $\bar{H}_{\bar{w}}(0, 0) = (0, \bar{\lambda}, 1)$, and solving for $Q(z, w, 0, 0)$ we find that

$$g = wQ(z, w, 0, 0) = w(1 + wr + \bar{\lambda}(\phi g + if^2)). \tag{4.4}$$

Similarly, differentiating (4.2) in $\bar{z}$ and evaluating at $\bar{z} = \bar{w} = 0$,

$$f = ws + zQ(z, w, 0, 0).$$ (4.5)

Along the first Segre set $\Sigma := \{w = 0\}$, we have

$$z - f + i\bar{\lambda}zf^2\big|_{\Sigma} = 0.$$

Solving for $f\big|_{\Sigma}$, we have that

$$f\big|_{\Sigma} = \frac{2z}{1 + \sqrt{1 - 4i\bar{\lambda}z^2}} = z + i\bar{\lambda}z^3 - 2\bar{\lambda}^2 z^5 + O\left(z^6\right),$$ (4.6)

where we choose a holomorphic branch of $\sqrt{1 - 4i\bar{\lambda}z^2}$ that is equal 1 at $z = 0$ so that the right-hand side is holomorphic at $z = 0$.

Applying the second-order differential operators $\partial_{\bar{z}}^j \partial_{\bar{w}}^{2-j}$, $j = 0, 1, 2$, to the mapping equation (4.2) and evaluating at $w = \bar{z} = \bar{w} = 0$, we obtain 3 linear equations of 3 unknowns r, s and ϕ along the first Segre set. Precisely, the following system holds along Σ.

$$\begin{pmatrix} 0 & 4iz & 1 \\ -z & 1 & 0 \\ i & 0 & i\bar{\lambda} \end{pmatrix} \cdot \begin{pmatrix} r \\ s \\ \phi \end{pmatrix} = f \begin{pmatrix} -\alpha f \\ i\alpha - \dfrac{\bar{\mu}}{2}f \\ 2\bar{\nu} - \dfrac{\bar{\sigma}}{2}f \end{pmatrix}.$$ (4.7)

This system is solvable since the coefficient 3×3-matrix in the left-hand side, denoted by D, is invertible with the inverse

$$D^{-1} = \frac{1}{1 - 4i\bar{\lambda}z^2} \begin{pmatrix} -\bar{\lambda} & -4i\bar{\lambda}z & 1 \\ -z\bar{\lambda} & -1 & z \\ 1 & 4iz & -4iz^2 \end{pmatrix}.$$

Explicitly,

$$\phi\big|_{\Sigma} = \frac{2z(\alpha + 4\bar{\nu}z)f + \left(4\bar{\sigma}z^2 + 2i\bar{\mu}z - \alpha\right)f^2}{1 - 4i\bar{\lambda}z^2}\Bigg|_{\Sigma},$$ (4.8)

$$r\big|_{\Sigma} = \frac{2(i\bar{\nu} - \alpha\bar{\lambda}z)f + (\alpha\bar{\lambda} + i\bar{\sigma} - 2i\bar{\lambda}\bar{\mu}z)f^2}{1 - 4i\bar{\lambda}z^2}\Bigg|_{\Sigma},$$ (4.9)

$$s\big|_{\Sigma} = \frac{(-\bar{\mu} + 2\alpha\bar{\lambda}z + 2i\bar{\sigma}z)f^2 + i(\alpha + 4\bar{\nu}z)f}{2(1 - 4i\bar{\lambda}z^2)}\Bigg|_{\Sigma}.$$ (4.10)

The proof is complete. □

Lemma 4.2. *If H is a map given in the normal form* (3.1), *then the following hold*

$$g_w(z, 0) = 1 + i\bar{\lambda}f(z, 0)^2,$$ (4.11)

$$f_w(z,0) = \frac{\bar{\lambda}zg_w(z,0)\phi(z,0) + zr(z,0) + s(z,0)}{\sqrt{1 - 4i\bar{\lambda}z^2}}, \tag{4.12}$$

$$g_{ww}(z,0) = 2\left(2i\bar{\lambda}f(z,0)f_w(z,0) + \bar{\lambda}g_w(z,0)\phi(z,0) + r(z,0)\right) \tag{4.13}$$

In particular, the following components of the third-order derivative of H are expressed in terms of its lower order derivatives

$$f^{(3,0)} = 6i\bar{\lambda}, \quad f^{(2,1)} = -\bar{\mu} + 8i\bar{\nu}, \quad g^{(3,0)} = 0, \tag{4.14}$$

$$g^{(2,1)} = 2i\bar{\lambda}, \quad g^{(1,2)} = 4i\bar{\nu}, \quad \phi^{(3,0)} = 6(8\bar{\nu} + 2i\bar{\mu}). \tag{4.15}$$

Proof. Differentiating (4.4) in w, setting $w = 0$, and substituting $g(z,0) = 0$, we find that

$$g_w(z,0) = Q(z,0,0,0) = 1 + i\bar{\lambda}f(z,0)^2.$$

Substituting $Q(z,w,0,0)$ from (4.4) into (4.5) and applying $\partial_w|_{w=0}$ we obtain

$$f_w(z,0) = \frac{\bar{\lambda}zg_w(z,0)\phi(z,0) + zr(z,0) + s(z,0)}{\sqrt{1 - 4i\bar{\lambda}z^2}}.$$

Differentiating (4.4) in w twice and evaluating along $w = 0$, we have

$$g_{ww}(z,0) = 2\left(2i\bar{\lambda}f(z,0)f_w(z,0) + \bar{\lambda}g_w(z,0)\phi(z,0) + r(z,0)\right).$$

Expanding the formula for $f(z,0)$ in Lemma 4.1 as Taylor series at $z = 0$ we have $f^{(3,0)} = 6i\bar{\lambda}$. $\quad\square$

Lemma 4.3. Let $H = (f, \phi, g)$ be a map given in the normal form (3.1). Then the following holds along Σ:

$$\Delta := \det \begin{pmatrix} -z & 0 & 0 & 2i\lambda f + 2i(\mu + 4i\nu)f^2 \\ i & -4z & 0 & 2\lambda + 2(\mu + 8i\nu)f - \bar{\phi}^{(2,1)}f^2 \\ 0 & 2i & -2z & 4\nu - 2\bar{f}^{(1,2)}f - \bar{\phi}^{(1,2)}f^2 - 2(i\bar{\mu} + 2\bar{\nu})\phi \\ 0 & 0 & 3i & i\bar{g}^{(0,3)} - 2\bar{f}^{(0,3)}f - \bar{\phi}^{(0,3)}f^2 - 6i\bar{\sigma}\phi \end{pmatrix} = 0. \tag{4.16}$$

Proof. Applying the third-order differential operators $\partial_{\bar{z}}^j \partial_{\bar{w}}^{3-j}$, $j = 0, 1, 2, 3$, to the mapping equation (4.2), evaluating at $w = \bar{z} = \bar{w} = 0$, and using Lemma 4.2, we obtain 4 linear equations of the following auxiliary holomorphic functions (or formal power series)

$$p(z,w) := Q_{\bar{z}\bar{z}}(z,w,0,0), \; t(z,w) := Q_{\bar{z}\bar{w}}(z,w,0,0), \; q(z,w) := Q_{\bar{w}\bar{w}}(z,w,0,0)$$

along the first Segre set. For instance, for $j = 0$, we can set $w = \bar{w} = 0$ in the mapping equation and using $g(z,0) = \bar{g}(\bar{z},0) = 0$, we have

$$f(z,0)\bar{f}(\bar{z},0) + \frac{1}{2}f(z,0)^2\bar{\phi}(\bar{z},0) + \frac{1}{2}\bar{f}(\bar{z},0)^2\phi(z,0) = \bar{z}zQ(z,0,\bar{z},0).$$

Differentiating this three times in $\bar{z}$ and setting $\bar{z} = 0$, we have

$$6i\lambda f(z,0) + 6(i\mu - 4\nu)f(z,0)^2 + 3zp(z,0) = 0.$$

Proceeding similarly for $j = 1, 2, 3$, we have three more equations. Omitting the detailed calculation, we conclude that

$$\begin{pmatrix} -z & 0 & 0 \\ i & -4z & 0 \\ 0 & 2i & -2z \\ 0 & 0 & 3i \end{pmatrix} \cdot \begin{pmatrix} p \\ t \\ q \end{pmatrix} = \begin{pmatrix} 2i\lambda f + 2i(\mu + 4i\nu)f^2 \\ 2\lambda + 2(\mu + 8i\nu)f - \bar{\phi}^{(2,1)}f^2 \\ 4\nu - 2\bar{f}^{(1,2)}f - \bar{\phi}^{(1,2)}f^2 - 2(i\bar{\mu} + 2\bar{\nu})\phi \\ i\bar{g}^{(0,3)} - 2\bar{f}^{(0,3)}f - \bar{\phi}^{(0,3)}f^2 - 6i\bar{\sigma}\phi \end{pmatrix}. \tag{4.17}$$

Observe that the 3×3-matrix formed by the last 3 rows of the coefficient matrix in the left-hand side of (4.17) is invertible. Thus, by the Kronecker-Capelli theorem, (4.17) has unique solution for p, q, and t along the first Segre set if and only if the determinant of the augmented matrix (4.16) vanishes identically along Σ. $\quad\square$

To reduce computational complexity, we shall divide into two cases, depending on whether $\lambda = 0$ or $\lambda \neq 0$. We first treat the case $\lambda \neq 0$.

4.1. Case 1: $\lambda \neq 0$

The purpose of this section is to compute H_w and H_{ww} along Σ for the case $\lambda \neq 0$.

Lemma 4.4. *Assume $H = (f, \phi, g)$ is given as in (3.1) and sends $\mathbb{H}^3$ into $\mathcal{X}$. If $\lambda \neq 0$, then*

$$H_w\big|_\Sigma = \left(0, \frac{2\lambda}{1 + \sqrt{1 - 4i\bar{\lambda}z^2}}, \frac{2}{1 + \sqrt{1 - 4i\bar{\lambda}z^2}}\right),$$

and

$$H_{ww}\big|_\Sigma = \left(\frac{8|\lambda|^2 z\sqrt{1 - 4i\bar{\lambda}z^2}}{\left(1 - 4i\bar{\lambda}z^2\right)\left(1 + \sqrt{1 - 4i\bar{\lambda}z^2}\right)^2}, 0, 0\right).$$

To prove this lemma, we first identify the coefficient μ, ν, and σ in the partial normal form (3.1) of the map.

Lemma 4.5. *Let H be a map given in the form (3.1). Assume that (4.1) and (4.16) hold. If $\lambda \neq 0$, then*

$$\mu = \nu = \sigma = 0. \tag{4.18}$$

Moreover, the following equalities between third-order derivatives at the origin hold:

$$\phi^{(0,3)} = 3\lambda f^{(1,2)}, \ \phi^{(1,2)} = 0, \ \phi^{(2,1)} = 2i|\lambda|^2, \ g^{(0,3)} = 3f^{(1,2)}. \tag{4.19}$$

Proof. We note that for $\lambda \neq 0$, we can write

$$f|_\Sigma = z\eta(z), \ f^2|_\Sigma = \frac{i}{\lambda}(1 - \eta(z)), \ \text{for} \ \eta(z) := 2\left(1 + \sqrt{1 - 4i\bar{\lambda}z^2}\right)^{-1},$$

where we choose the holomorphic branch of the squared root having value 1 at $z = 0$. Plugging these into the formula for $\phi|_\Sigma$ in (4.8), we find that

$$\phi|_\Sigma = u(z) + v(z)\eta(z)$$

for some rational functions $u(z)$ and $v(z)$ holomorphic at $z = 0$. Expanding the determinant in (4.16) along the first row, we have

$$\Delta = -z\Delta_{1,1} - (2i\lambda f + 2i(\mu + 4i\nu)f^2)\Delta_{1,4},$$

where $\Delta_{1,1}$ and $\Delta_{1,4}$ are the corresponding minors. Observe that $\Delta_{1,4} = -6i$ and, by the formulas for f, f^2 and ϕ along the first Segre set above, $\Delta_{1,1} = C(z) + D(z)\eta(z)$ for some rational functions $C(z)$ and $D(z)$ holomorphic at the origin. Therefore,

$$\Delta = -\frac{12i}{\bar{\lambda}}(\mu + 4i\nu) - zC(z) + \left(\frac{12i}{\bar{\lambda}}(\mu + 4i\nu) - 12\lambda z - zD(z)\right)\eta(z).$$

Since $\eta(z)$ is irrational when $\lambda \neq 0$, the vanishing of Δ is equivalent to

$$\nu = \frac{i\mu}{4}, \quad C(z) = 0, \quad D(z) = -12\lambda.$$

The last identity is expressed explicitly as

$$\det \begin{pmatrix} -4z & 0 & 2\lambda - 2\mu f - \bar{\phi}^{(2,1)}f^2 \\ 2i & -2z & i\mu - 2\bar{f}^{(1,2)}f - \bar{\phi}^{(1,2)}f^2 - i\bar{\mu}\phi \\ 0 & 3i & i\bar{g}^{(0,3)} - 2\bar{f}^{(0,3)}f - \bar{\phi}^{(0,3)}f^2 - 6i\bar{\sigma}\phi \end{pmatrix} = -12\lambda\eta(z).$$

Expanding the determinant along the first row, canceling the right-hand side, and dividing the resulting equation by -2, we find that

$$2f\left(4z^2\bar{f}^{(0,3)} + 6iz\bar{f}^{(1,2)} - 3\mu\right) + f^2\left(4z^2\bar{\phi}^{(0,3)} + 6iz\bar{\phi}^{(1,2)} - 3\bar{\phi}^{(2,1)}\right)$$
$$+ 6\phi\left(4i\bar{\sigma}z^2 - \bar{\mu}z\right) - 4iz^2\bar{g}^{(0,3)} + 6\mu z + 6\lambda(1 - \eta) = 0. \tag{4.20}$$

Substituting f and ϕ along Σ from (4.1) and collecting the term $\sqrt{1 - 4i\bar{\lambda}z^2}$, we can rewrite the equation as

$$\frac{A(z) + B(z)\sqrt{1 - 4i\bar{\lambda}z^2}}{\left(4\bar{\lambda}z^2 + i\right)\left(1 + \sqrt{1 - 4i\bar{\lambda}z^2}\right)^2} = 0,$$

where A and B are polynomials in z. The vanishing of Δ implies that A and B must vanish identically. Explicitly,

$$B(z) = 4z^2\{8\left(2\bar{\lambda}\bar{f}^{(0,3)} + 3i\bar{\mu}\bar{\sigma}\right)z^3 + 2\left(12i\bar{\lambda}\bar{f}^{(1,2)} - 4i\bar{\lambda}\bar{g}^{(0,3)} - 12\alpha\bar{\sigma} - 3\bar{\mu}^2\right)z^2$$

$$+2\left(2i\bar{f}^{(0,3)} - 3i\alpha\bar{\mu}\right)z + 2\bar{g}^{(0,3)} - 6\bar{f}^{(1,2)}\}.$$

Since $B(z)$ must vanish identically, equating the coefficient of the lowest degree in z, that of z^2, yields

$$\bar{g}^{(0,3)} = 3\bar{f}^{(1,2)}.$$

The vanishing of the coefficient of z^3 yields

$$\bar{f}^{(0,3)} = 3\frac{\alpha\bar{\mu}}{2}.$$

These together with the vanishing of the terms of degree 4 in z yields

$$4\alpha\bar{\sigma} + \bar{\mu}^2 = 0.$$

Combining these three equations with the vanishing of the degree 5 term yields

$$\bar{\mu}(\alpha\bar{\lambda} + i\bar{\sigma}) = 0.$$

On the other hand, we have

$$A(z) = 4z^2\{16z^4\left(6\bar{\sigma}^2 + \bar{\phi}^{(0,3)}\bar{\lambda} - \bar{\lambda}^2\bar{g}^{(0,3)}\right) + 8z^3\left(2\bar{\lambda}\bar{f}^{(0,3)} + 3i\bar{\lambda}\bar{\phi}^{(1,2)} - 3i\bar{\lambda}^2\mu - 6i\bar{\mu}\bar{\sigma}\right)$$

$$+2z^2\left(12i\bar{\lambda}\bar{f}^{(1,2)} - 6i\bar{\lambda}\bar{g}^{(0,3)} - 6\bar{\lambda}\bar{\phi}^{(2,1)} + 2i\bar{\phi}^{(0,3)} - 12i\lambda\bar{\lambda}^2 + 3\bar{\mu}^2\right)$$

$$+z\left(4i\bar{f}^{(0,3)} - 6\bar{\phi}^{(1,2)} + 6\bar{\lambda}\mu\right) - 6\bar{f}^{(1,2)} + 2\bar{g}^{(0,3)} - 3i\bar{\phi}^{(2,1)} + 6|\lambda|^2\}.$$

Since A must vanish identically, equating the coefficient of z^2 to zero and using the relation $\bar{g}^{(0,3)} = 3\bar{f}^{(1,2)}$ obtained above, we find that

$$\bar{\phi}^{(2,1)} = -2i|\lambda|^2.$$

Putting these together we obtain

$$A(z) = 4z^2\{16z^4\left(6\bar{\sigma}^2 + \bar{\lambda}(\bar{\phi}^{(0,3)} - 3\bar{\lambda}\bar{f}^{(1,2)})\right) + 8z^3\left(3i\bar{\lambda}\bar{\phi}^{(1,2)} - 3i\bar{\lambda}^2\mu - 6i\bar{\mu}\bar{\sigma} + 3\alpha\bar{\lambda}\bar{\mu}\right)$$

$$+2z^2\left(2i(\bar{\phi}^{(0,3)} - 3\bar{\lambda}\bar{f}^{(1,2)}) + 3\bar{\mu}^2\right) + 6z\left(i\alpha\bar{\mu} + \bar{\lambda}\mu - \bar{\phi}^{(1,2)}\right)\}.$$

Equating the coefficient of z^3 to zero we find that

$$\bar{\phi}^{(1,2)} = \bar{\lambda}\mu + i\alpha\bar{\mu}.$$

Plugging this back to the formula for $A(z)$, we find that the coefficient of z^5 is a multiple of $\bar{\sigma}\bar{\mu}$. Equating this to be zero, we find that either $\bar{\sigma} = 0$ of $\bar{\mu} = 0$. But in either case, the vanishing of the coefficients of z^6 and z^4 in $A(z)$ together imply that $\bar{\sigma} = \bar{\mu} = 0$. Moreover,

$$\bar{\phi}^{(0,3)} = 3\bar{\lambda}\bar{f}^{(1,2)}.$$

That $\bar{\phi}^{(1,2)} = 0$ is obvious since $\mu = 0$.   □

Lemma 4.6. *Let H be a map given by* (3.1). *If $\lambda \neq 0$, then the auxiliary functions p, t, q along the first Segre set Σ are given by*

$$p(z,0) = 2i\lambda\eta(z), \ t(z,0) = 0, \ q(z,0) = \bar{f}^{(1,2)}\eta(z).$$

Proof. Plugging (4.18) and (4.19) from Lemma 4.5 into the system (4.17), we have

$$\begin{pmatrix} i & -4z & 0 \\ 0 & 2i & -2z \\ 0 & 0 & 3i \end{pmatrix} \cdot \begin{pmatrix} p \\ t \\ q \end{pmatrix} = \begin{pmatrix} 2\lambda + 2i|\lambda|^2 f^2 \\ -2\bar{f}^{(1,2)}f \\ 3i\bar{f}^{(1,2)} - 3\bar{\lambda}\bar{f}^{(1,2)}f^2 \end{pmatrix} = \eta(z) \begin{pmatrix} 2\lambda \\ -2\bar{f}^{(1,2)}z \\ 3i\bar{f}^{(1,2)} \end{pmatrix}.$$

This can be solved easily to obtain the desired formulas for $p, t,$ and q.   □

Lemma 4.7. *Assume that $\lambda \neq 0$. Then $\alpha = 0$ and*

$$\phi^{(3,1)} = \phi^{(2,2)} = f^{(0,4)} = 0, \ g^{(0,4)} = 4f^{(1,3)},$$

$$g^{(1,3)} = \frac{3}{2}f^{(2,2)}, \phi^{(1,3)} = \frac{3}{2}\bar{\lambda}f^{(2,2)}, \phi^{(0,4)} = 4\bar{\lambda}f^{(1,3)}.$$

Proof. We introduce the following auxiliary holomorphic functions

$$k(z,w) = Q_{\bar{z}\bar{z}\bar{z}}(z,w,0,0), \ l(z,w) = Q_{\bar{z}\bar{z}\bar{w}}(z,w,0,0),$$

$$m(z,w) = Q_{\bar{z}\bar{w}\bar{w}}(z,w,0,0), \ n(z,w) = Q_{\bar{w}\bar{w}\bar{w}}(z,w,0,0).$$ (4.21)

We proceed similarly to the proof of Lemma 4.3 as follows. Applying the differential operator $\partial_{\bar{z}}^j \partial_{\bar{w}}^{4-j}$, $j = 0, 1, 2, 3, 4$, to the mapping equation (4.2) and evaluating at $w = \bar{z} = \bar{w} = 0$, we obtain an overdetermined system of 5 linear equations of 4 unknowns that must be satisfied by k, l, m, n restricted to the first Segre set $\Sigma = \{w = 0\}$. Explicitly, the following holds when $w = 0$:

$$
\begin{pmatrix}
-z & 0 & 0 & 0 \\
i & -6z & 0 & 0 \\
0 & 2i & -4z & 0 \\
0 & 0 & 3i & -2z \\
0 & 0 & 0 & 4i
\end{pmatrix} \cdot
\begin{pmatrix}
k \\
l \\
m \\
n
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
6i\lambda(2\alpha f^2 + \phi) \\
30\alpha\lambda f - \bar{\phi}^{(3,1)} f^2 \\
-2\bar{f}^{(2,2)} f - \bar{\phi}^{(2,2)} f^2 - 4\bar{f}^{(1,2)}\phi + \alpha^2\phi - 4|\lambda|^2\phi - 2i\bar{\phi}^{(2,1)}\phi - 4i\alpha\lambda \\
-2\bar{f}^{(1,3)} f - \bar{\phi}^{(1,3)} f^2 - 2\bar{f}^{(0,3)}\phi + i\bar{g}^{(1,3)} - 3i\bar{\phi}^{(1,2)}\phi \\
-2\bar{f}^{(0,4)} f - \bar{\phi}^{(0,4)} f^2 - 4i\phi\left(\bar{\lambda}\bar{g}^{(0,3)} + \bar{\phi}^{(0,3)}\right) + i\bar{g}^{(0,4)}
\end{pmatrix}
\tag{4.22}
$$

The 4×4-matrix formed by the last 4 rows of the coefficient matrix is invertible (its determinant is 24). Thus, by Kronecker-Capelli theorem again, the solvability of this system is equivalent to the vanishing (identically along Σ) of the determinant of the augmented 5×5-matrix which we denoted by Δ.

Since $\mu = \nu = \sigma = 0$, we have from (4.8)

$$
\phi\big|_{\Sigma} = \alpha \frac{2zf - f^2}{1 - 4i\bar{\lambda}z^2}\bigg|_{\Sigma}.
\tag{4.23}
$$

Therefore, one can write

$$
6i\lambda(2\alpha f^2 + \phi) = -\frac{6\alpha\lambda\left(8\bar{\lambda}z^2 + i\right)}{\bar{\lambda}\left(4\bar{\lambda}z^2 + i\right)} + E(z)\eta(z),
$$

where $E(z)$ is a rational function of z holomorphic at $z = 0$. Expanding the determinant Δ of the 5×5 augmented matrix along the first row, we have

$$
\Delta = -z\Delta_{1,1} + (6i\lambda(2\alpha f^2 + \phi))\Delta_{1,5},
$$

where $\Delta_{1,1}$ and $\Delta_{1,5}$ are the corresponding minors. Arguing as before, we have

$$
\Delta_{1,1} = F(z) + G(z)\eta(z),
$$

where $F(z)$ and $G(z)$ are also rational in z and holomorphic at $z = 0$, while $\Delta_{1,5} = 24$. Plugging this back into the formula for Δ, we have

$$-zF(z) - 24\frac{6\alpha\lambda\left(8\bar{\lambda}z^2 + i\right)}{\bar{\lambda}\left(4\bar{\lambda}z^2 + i\right)} + \eta(z)(24E(z) - zG(z)).$$

By irrationality of $\eta(z)$, Δ vanishes identically iff

$$-zF(z) - 24\frac{6\alpha\lambda\left(8\bar{\lambda}z^2 + i\right)}{\bar{\lambda}\left(4\bar{\lambda}z^2 + i\right)} = 0, \quad 24E(z) - zG(z) = 0.$$

Setting $z = 0$ in the first equation, we find that $\alpha = 0$, as desired. $\quad\square$

Since the corresponding solvability conditions are satisfied, we can solve for k, l, m, and n from

$$\begin{pmatrix} i & -6z & 0 & 0 \\ 0 & 2i & -4z & 0 \\ 0 & 0 & 3i & -2z \\ 0 & 0 & 0 & 4i \end{pmatrix} \cdot \begin{pmatrix} k \\ l \\ m \\ n \end{pmatrix} = \eta(z) \begin{pmatrix} 0 \\ -2\bar{f}^{(2,2)}z \\ -2\bar{f}^{(1,3)}z - \dfrac{3i}{2}\bar{f}^{(2,2)} \\ 4i\bar{f}^{(1,3)} \end{pmatrix} \tag{4.24}$$

to obtain the following identity along Σ:

$$k(z,0) = m(z,0) = 0, \quad m(z,0) = \frac{1}{2}\bar{f}^{(2,2)}\eta(z), \quad n(z,0) = \bar{f}^{(1,3)}\eta(z). \tag{4.25}$$

Proof of Lemma 4.4. Differentiating the mapping equation with respect to $\bar{z}$ and $\bar{w}$ and combining the two equations, we obtain

$$z + i\bar{\lambda}zf(z,w)^2 - f(z,w) + \bar{\lambda}zg(z,w)\phi(z,w) + wzr(z,w) + ws(z,w) = 0. \tag{4.26}$$

Differentiating this with respect to w and setting $w = 0$ yields

$$\left.(2i\bar{\lambda}zf - 1)f_w + zr + s\right|_{\Sigma} = 0.$$

Plugging $\alpha = \sigma = \mu = \nu = 0$ into (4.7), we easily find that $r = s = 0$ along $w = 0$. Thus

$$\left.f_w\right|_{\Sigma} = 0.$$

Applying $\partial_{\bar{z}}^j\partial_{\bar{w}}^{2-j}\partial_w$, $j = 0, 1, 2$, to the mapping equation and evaluating along Σ, we obtain

$$\begin{pmatrix} 0 & 4iz & 1 \\ -z & 1 & 0 \\ i & 0 & i\bar{\lambda} \end{pmatrix} \cdot \begin{pmatrix} r_w \\ s_w \\ \phi_w \end{pmatrix} = \frac{i}{2} \begin{pmatrix} p \\ t \\ q \end{pmatrix}.$$

Plugging the formulas for p, t, and q along Σ from Lemma 4.6, we find that

$$\phi_w\big|_\Sigma = \frac{\eta(z)\left(\lambda - 2iz^2\bar{f}^{(1,2)}\right)}{1 - 4i\bar{\lambda}z^2}.$$

We claim that

$$f^{(1,2)} = 2|\lambda|^2.$$

To show this, we differentiate (4.26) twice in w and setting $w = 0$, using $\phi = 0$, $f_w = 0$, $g_w = 1 + \bar{\lambda}f^2$ along Σ to obtain

$$f_{ww}\big|_\Sigma = -\frac{4z\left(2|\lambda|^2 - \bar{f}^{(1,2)}\left(1 + \sqrt{1 - 4i\bar{\lambda}z^2}\right)\right)}{\left(1 - 4i\bar{\lambda}z^2\right)\left(1 + \sqrt{1 - 4i\bar{\lambda}z^2}\right)^2}.$$

Differentiating this equation in z and evaluating at $z = 0$ we find that

$$f^{(1,2)} = -2|\lambda|^2 + 2\bar{f}^{(1,2)}.$$

Hence $f^{(1,2)} = 2|\lambda|^2$ and the claim follows. Thus

$$f_{ww}\big|_\Sigma = \frac{8|\lambda|^2 z\sqrt{1 - 4i\bar{\lambda}z^2}}{\left(1 - 4i\bar{\lambda}z^2\right)\left(1 + \sqrt{1 - 4i\bar{\lambda}z^2}\right)^2}.$$

In the next step, we solve for p_w, t_w and q_w along the first Segre set. To this end, we apply $\partial_{\bar{z}}^j\partial_{\bar{w}}^{3-j}\partial_w$, $j = 0, 1, 2, 3$, to the mapping equation (4.2) and evaluate along $\bar{z} = \bar{w} = w = 0$ to obtain

$$\begin{pmatrix} -6z & 0 & 0 \\ i & -4z & 0 \\ 0 & 2i & -2z \\ 0 & 0 & 3i \end{pmatrix} \cdot \begin{pmatrix} p_w \\ t_w \\ q_w \end{pmatrix} = i \begin{pmatrix} k \\ l \\ m \\ n \end{pmatrix}$$

when $w = 0$. The solvability of this overdetermined system, via Kronecker-Capelli theorem, gives

$$k(z, 0) - 6izl(z, 0) - 12z^2 m(z, 0) + 8iz^3 n(z, 0) = 0,$$

which, from the formula (4.25) for k, l, m, and l along Σ, is equivalent to

$$12iz\left(4z\bar{f}^{(1,3)} + 3i\bar{f}^{(2,2)}\right)f\big|_\Sigma = 0,$$

or equivalently,

$$\bar{f}^{(1,3)} = 0, \ \bar{f}^{(2,2)} = 0.$$

Hence, from (4.25), we find that $k = l = m = n = 0$ along Σ and consequently $p_w = t_w = q_w = 0$ along Σ.

Finally, applying $\partial_z^j \partial_{\bar{w}}^{2-j} \partial_w^2$, $j = 0, 1, 2$, and evaluating along $w = \bar{z} = \bar{w}$, we have

$$
\begin{pmatrix} 0 & 4iz & 1 \\ -z & 1 & 0 \\ i & 0 & i\bar{\lambda} \end{pmatrix} \cdot \begin{pmatrix} r_{ww} \\ s_{ww} \\ \phi_{ww} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

and we find that

$$
\phi_{ww}\big|_\Sigma = 0.
$$

The proof is complete. $\square$

We sum up and use the notation $O(5)$, which denotes power series centered at the origin whose terms are of total degree greater than or equal to 5.

Lemma 4.8. *If $\lambda \neq 0$, then*

$$
H(z, w) = (1 + iz^2 + w^2)(z, w, w) + O(5),
$$

and

$$
H(z, 0) = \left(\frac{2z}{1 + \sqrt{1 - 4iz^2}}, 0, 0 \right),
$$

$$
H_w(z, 0) = \left(0, \frac{2}{1 + \sqrt{1 - 4iz^2}}, \frac{2}{1 + \sqrt{1 - 4iz^2}} \right),
$$

$$
H_{ww}(z, 0) = \left(\frac{8z\sqrt{1 - 4iz^2}}{(1 - 4iz^2)(1 + \sqrt{1 - 4iz^2})^2}, 0, 0 \right).
$$

In the next step we want to determine the map along the second Segre set, parametrized by $(z, \bar{z}) \mapsto (z, 2iz\bar{z})$. To this end we evaluate the mapping equation and its first and second CR derivative at $\bar{w} = 0$. More precisely, denote by L the global $(1, 0)$-vector field given by

$$
L = \frac{\partial}{\partial z} + 2i\bar{z}\frac{\partial}{\partial w},
$$

and $\overline{L}$ its conjugate. Observe that $L\varrho = 0$ on $\mathbb{C}^2$ (not just along $\mathbb{H}^3$). We consider the system

$$
L^k \rho'\big(H(z, \bar{w} + 2iz\bar{z}), \overline{H(z, w)}\big)\big|_{\bar{w}=0} = 0, \quad k \in \{0, 1, 2\}.
$$

Using the fact that $\phi(z,0) = g(z,0) = f_w(z,0) = \phi_{ww}(z,0) = g_{ww}(z,0) = 0$, the above system reduces to the following, where we skip evaluation along $\bar{\Sigma}$ in $\bar{H}$ and its derivatives and along the second Segre set:

$$2i\bar{f}f + i\bar{f}^2\phi - g = 0, \tag{4.27}$$

$$iz\bar{\phi}_{\bar{w}}f^2 - \bar{f}_{\bar{z}}f - (\bar{f}\bar{f}_{\bar{z}} - z\bar{\phi}_{\bar{w}}g)\phi + z\bar{g}_{\bar{w}} = 0, \tag{4.28}$$

$$2iz\bar{\phi}_{\bar{z}\bar{w}}f^2 + (4z^2\bar{f}_{\bar{w}\bar{w}} - \bar{f}_{\bar{z}\bar{z}})f + ((4z^2\bar{f}_{\bar{w}\bar{w}} - \bar{f}_{\bar{z}\bar{z}})\bar{f} - \bar{f}_{\bar{z}}^2 + 4iz^2\bar{g}_{\bar{w}}\bar{\phi}_{\bar{w}})\phi \tag{4.29}$$
$$+ 2z\bar{\phi}_{\bar{z}\bar{w}}\phi g + 2z\bar{g}_{\bar{z}\bar{w}} = 0.$$

Combining (4.28) and (4.29) and eliminating f^2 lead to an equation of the form:

$$A + Bf + (C + \bar{f}B)\phi = 0, \tag{4.30}$$

where

$$A = 2z(\bar{g}_{\bar{z}\bar{w}}\bar{\phi}_{\bar{w}} - \bar{g}_{\bar{w}}\bar{\phi}_{\bar{z}\bar{w}}),$$
$$B = \bar{\phi}_{\bar{w}}\left(4z^2\bar{f}_{\bar{w}\bar{w}} - \bar{f}_{\bar{z}\bar{z}}\right) + 2\bar{f}_{\bar{z}}\bar{\phi}_{\bar{z}\bar{w}},$$
$$C = -\bar{f}_{\bar{z}}^2\bar{\phi}_{\bar{w}} + 4iz^2\bar{g}_{\bar{w}}\bar{\phi}_{\bar{w}}^2.$$

It can be checked that $A = 0$ and $-2i\bar{z}(C + \bar{f}B) = B$, hence (4.30) implies $\phi = 2i\bar{z}f$, such that (4.27) shows $g = \phi$. The remaining equation is given by

$$iz(1 + 4i\bar{z}^2)f^2 - f + z = 0.$$

The solution compatible with (4.6) is

$$f(z, 2iz\bar{z}) = \frac{1 - \sqrt{1 - 4iz^2(1 + 4i\bar{z}^2)}}{z(2i - 8\bar{z}^2)},$$

setting $\bar{z} = w/(2iz)$ gives

$$f(z,w) = \frac{z(1 - \sqrt{1 - 4w^2 - 4iz^2})}{2(w^2 + iz^2)}.$$

If we rewrite the denominator via

$$2(w^2 + iz^2) = \frac{1}{2}(1 - \sqrt{1 - 4w^2 - 4iz^2})(1 + \sqrt{1 - 4w^2 - 4iz^2}),$$

we obtain

$$f(z,w) = \frac{2z}{1 + \sqrt{1 - 4w^2 - 4iz^2}},$$

which gives the desired map ι in Theorem 1.2 and finishes the case $\lambda \neq 0$.

4.2. Case 2: $\lambda = 0$

In this case, Lemma 4.1 implies that

$$H\big|_\Sigma = \left(z, \alpha z^2 + 2z^3(4\bar\nu + i\bar\mu) + 4\bar\sigma z^4, 0\right).$$

We shall show that μ, ν, and σ must be zero.

Lemma 4.9. *Assume that $\lambda = 0$, then $\sigma = 0$ and the following holomorphic identity holds:*

$$4z^3 g - 4z^2 wf + w^2 z\phi - w^2 \Upsilon(z,w)\left(w(\mu + 4i\nu) - \alpha z\right) = 0, \qquad (4.31)$$

where $\Upsilon := g\phi + if^2$.

Proof. Setting $w = \bar w + 2iz\bar z$ in the mapping equation (4.2), we rewrite it in the parametrized form:

$$(1 - \phi(z, \bar w + 2iz\bar z)\bar\phi(\bar z, \bar w))(g(z, \bar w + 2iz\bar z) - \bar g(\bar z, \bar w)) - 2if(z, \bar w + 2iz\bar z)\bar f(\bar z, \bar w)$$
$$-if(z, \bar w + 2iz\bar z)^2 \bar\phi(\bar z, \bar w) - i\bar f(\bar z, \bar w)^2 \phi(z, \bar w + 2iz\bar z) = 0 \quad (4.32)$$

which is satisfied for $z, \bar z, \bar w$ in a neighborhood of the origin in $\mathbb{C}^3$ or as identity of formal power series.

Setting $\bar w = 0$ and substituting $\bar f(\bar z, 0) = \bar z$, $\bar g(\bar z, 0) = 0$, and $\bar\phi(\bar z, 0) = \alpha\bar z^2 + 2\bar z^3(4\nu - i\mu) + 4\sigma\bar z^4$, we have

$$g(z, 2i|z|^2) - \Upsilon(z, 2i|z|^2)\left(\alpha\bar z^2 + 2\bar z^3(4\nu - i\mu) + 4\sigma\bar z^4\right)$$
$$- 2i\bar z f(z, 2i|z|^2) - i\bar z^2 \phi(z, 2i|z|^2) = 0.$$

From this, we obtain a holomorphic equation for f, g, and ϕ. Formally, we substitute $\bar z = w/(2iz)$, to get

$$g(z, w) - \Upsilon(z, w)\left(\frac{w^2\left(\sigma w^2 + wz(\mu + 4i\nu) - \alpha z^2\right)}{4z^4}\right)$$
$$- \left(\frac{w}{z}\right)f(z, w) + \left(\frac{w^2}{4z^2}\right)\phi(z, w) = 0.$$

Clearing the denominator $4z^4$ we have

$$4z^4 g - w^2\Upsilon(z, w)\left(\sigma w^2 + wz(\mu + 4i\nu) - \alpha z^2\right) - 4z^3 wf + w^2 z^2 \phi = 0. \qquad (4.33)$$

This holds as an identity of germs of holomorphic functions at the origin if H is holomorphic. If H is only assumed to be smooth or formal CR map, then the above holds as

formal power series. Indeed, if $G(z, w)$ denotes the left-hand side, then the computation above shows that

$$G(z, 2iz\bar{z}) = 0$$

as an identity of formal power series of z and $\bar{z}$. Since the Segre map $(z, \bar{z}) \mapsto (z, 2iz\bar{z})$ is of generic full rank (in other words, $\mathbb{H}^3$ is minimal, cf. [3]), we can apply a well-known result (as stated and proved in [3, Proposition 5.3.5]) to conclude that $G(z, w)$ must be zero in $\mathbb{C}[[z, w]]$.

Setting $z = 0$ in (4.33), we obtain

$$\sigma w^4 \Upsilon(0, w) = 0.$$

Thus, either $\sigma = 0$, or $\Upsilon(0, w) = 0$. If the latter statement holds, we differentiate the identity three times in w and evaluate at the origin to get

$$\sigma = \frac{1}{2}\phi_{ww} = \frac{1}{2}(g\phi + if^2)_{www} = \Upsilon_{www} = 0.$$

Hence in all cases it holds that $\sigma = 0$ and (4.33) reduces to (4.31), as desired. Setting $z = 0$ in the last equation, we see that either $\mu + 4i\nu = 0$, or $\Upsilon(0, w) = 0$. $\quad\square$

Lemma 4.10. *If $\lambda = 0$, then $\sigma = 0$ and*

$$\phi^{(0,3)} = 6(6\mu\nu - i\mu^2 + 8i\nu^2), \tag{4.34}$$

$$\phi^{(1,2)} = -\frac{4}{3}if^{(0,3)} + 2\alpha(i\mu - 2\nu), \tag{4.35}$$

$$\phi^{(2,1)} = \frac{4}{3}ig^{(0,3)} - 4if^{(1,2)}, \tag{4.36}$$

$$\phi^{(3,0)} = 12(4\bar{\nu} + i\bar{\mu}). \tag{4.37}$$

Proof. By Lemma 4.3 and the assumption $\lambda = 0$ we have

$$\Delta := \det \begin{pmatrix} -z & 0 & 0 & 2i(\mu + 4i\nu)f^2 \\ i & -4z & 0 & 2(\mu + 8i\nu)f - \bar{\phi}^{(2,1)}f^2 \\ 0 & 2i & -2z & 4\nu - 2\bar{f}^{(1,2)}f - \bar{\phi}^{(1,2)}f^2 - 2(i\bar{\mu} + 2\bar{\nu})\phi \\ 0 & 0 & 3i & i\bar{g}^{(0,3)} - 2\bar{f}^{(0,3)}f - \bar{\phi}^{(0,3)}f^2 - 6i\bar{\sigma}\phi \end{pmatrix} = 0, \tag{4.38}$$

along Σ. On the other hand, when $\lambda = 0$ we have

$$H\big|_\Sigma = \left(z, \alpha z^2 + 2z^3(4\bar{\nu} + i\bar{\mu}) + 4\bar{\sigma}z^4, 0\right).$$

If $\sigma \neq 0$, then $\phi\big|_\Sigma$ has degree exactly equal 4. Observe that $f\big|_\Sigma = z$ and hence the last column in the matrix on the right of (4.38) has degree at most 4 in z. We claim that

Δ has degree exactly 7. In fact, expanding the determinant dropping all terms of degree less than 7 we find that the degree 7 term in Δ is $192i\bar{\sigma}^2z^7$. Thus, the vanishing of Δ along Σ implies that $\sigma = 0$.

With $\sigma = 0$, substituting $f(z,0) = z$ and $\phi(z,0) = \alpha z^2 + 2z^3(4\bar{\nu} + i\bar{\mu})$ into (4.38) and equating coefficients of z to be zero, we can easily find the formulas as above. □

Lemma 4.11. *If $\lambda = 0$, then $\mu = -3i\nu$ or $\mu = -4i\nu$.*

Proof. Applying the differential operator $\partial_{\bar{z}}^4$ to the mapping equation (4.2), evaluating at $\bar{z} = \bar{w} = 0$, and using $\phi^{(4,0)} = 96\bar{\sigma} = 0$, we find that $k(z,w) = 0$, where k is the auxiliary function defined in (4.21). Next, applying the differential operator $\partial_{\bar{w}}^j\partial_{\bar{z}}^{4-j}$, $j = 1, 2, 3, 4$, to the mapping equation (4.2) and evaluating at $w = \bar{z} = \bar{w} = 0$, we obtain an overdetermined system of 4 linear equations of 3 unknowns that must be satisfied by l, m, n restricted to the first Segre set $\Sigma = \{w = 0\}$. Explicitly, the Kronecker-Capelli theorem implies that, when $w = 0$,

$$\det \begin{pmatrix} -6z & 0 & 0 & -z^2\bar{\phi}^{(3,1)} - 6\mu\phi \\ 2i & -4z & 0 & (\alpha^2 - 2i\bar{\phi}^{(2,1)} - 4\bar{f}^{(1,2)})\phi - 2z\bar{f}^{(2,2)} - z^2\bar{\phi}^{(2,2)} \\ 0 & 3i & -2z & (6i\alpha\bar{\nu} - 3i\bar{\phi}^{(1,2)} - 2\bar{f}^{(0,3)})\phi - 2z\bar{f}^{(1,3)} + i\bar{g}^{(1,3)} - z^2\bar{\phi}^{(1,3)} \\ 0 & 0 & 4i & -4\phi(6\bar{\nu}^2 - i\bar{\phi}^{(0,3)}) - 2z\bar{f}^{(0,4)} + i\bar{g}^{(0,4)} - z^2\bar{\phi}^{(0,4)} \end{pmatrix} = 0.$$

Denote this determinant by Δ. Observe that the entries in the last column of the matrix are polynomials in z of degree at most 3 and hence Δ have degree at most 6. Moreover, all the terms in the determinant expansion of Δ have degree at most 5 except the product of the diagonal entries. Thus, collecting the term of degree 6 in z we have

$$\Delta = 384z^6(\bar{\mu} - 4i\bar{\nu})\left(\bar{\phi}^{(0,3)} - 6i\bar{\nu}^2\right) + \cdots$$

where the dots represent terms of degree 5 or less in z. Thus, the vanishing of Δ implies that either $\mu = -4i\nu$ or $\bar{\phi}^{(0,3)} - 6i\bar{\nu}^2 = 0$. In the latter case, we use Lemma 4.10 to conclude that $\mu = -3i\nu$, as desired. □

Lemma 4.12. *If $\lambda = 0$ and $\mu = -4i\nu$, then $\mu = \nu = 0$.*

Proof. Since $\mu = -4i\nu$, we have $\phi(z,0) = \alpha z^2$, and $\bar{\phi}^{(0,3)} = 0$, $\bar{\phi}^{(1,2)} = -\frac{4}{3}i\bar{f}^{(0,3)} + 4\alpha\bar{\nu}$, $\bar{\phi}^{(2,1)} = -\frac{4}{3}i\bar{g}^{(0,3)} + 4i\bar{f}^{(1,2)}$. Thus

$$\det \begin{pmatrix} -6z & 0 & 0 & (24i\alpha\nu - \bar{\phi}^{(3,1)})z^2 \\ 2i & -4z & 0 & (\alpha^3 - \frac{8}{3}\alpha\bar{g}^{(0,3)} + 4\alpha\bar{f}^{(1,2)} - \bar{\phi}^{(2,2)})z^2 - 2z\bar{f}^{(2,2)} \\ 0 & 3i & -2z & (2\alpha\bar{f}^{(0,3)} - 6i\alpha^2\bar{\nu} - \bar{\phi}^{(1,3)})z^2 - 2z\bar{f}^{(1,3)} + i\bar{g}^{(1,3)} \\ 0 & 0 & 4i & -(24\alpha\bar{\nu}^2 + \bar{\phi}^{(0,4)})z^2 - 2z\bar{f}^{(0,4)} + i\bar{g}^{(0,4)} \end{pmatrix} = 0.$$

Expanding the determinant on the left, we obtain a degree 5 polynomial in z. Equating the terms of degree 5, we obtain

$$\bar{\phi}^{(0,4)} = -24\alpha\bar{\nu}^2.$$

Similarly, equating coefficients of degree 4 terms, we have

$$\bar{\phi}^{(1,3)} = 2\alpha\bar{f}^{(0,3)} + i\left(\bar{f}^{(0,4)} - 6\alpha^2\bar{\nu}\right).$$

Equating the terms of degree 3 yields

$$\bar{\phi}^{(2,2)} = 4\alpha\bar{f}^{(1,2)} - \frac{2}{3}i\left(\bar{g}^{(0,4)} - 4\bar{f}^{(1,3)}\right) - \frac{8}{3}\alpha\bar{g}^{(0,3)} + \alpha^3.$$

Equating the terms of second degree

$$\bar{\phi}^{(3,1)} = 6i\bar{f}^{(2,2)} - 4i\bar{g}^{(1,3)} - 6\alpha\mu.$$

With these conditions satisfied, the system is solvable possessing a unique solution, which can be found by solving the system

$$\begin{pmatrix} 2i & -4z & 0 \\ 0 & 3i & -2z \\ 0 & 0 & 4i \end{pmatrix} \cdot \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} -2z\bar{f}^{(2,2)} + \frac{2}{3}iz^2\left(\bar{g}^{(0,4)} - 4\bar{f}^{(1,3)}\right) \\ -iz^2\bar{f}^{(0,4)} - 2z\bar{f}^{(1,3)} + i\bar{g}^{(1,3)} \\ -2z\bar{f}^{(0,4)} + i\bar{g}^{(0,4)} \end{pmatrix} \qquad (4.39)$$

We observe that, when restricted to Σ, n is linear, m is at most quadratic, and l is at most cubic in z (actually, the unique solutions l, m, and n along Σ are linear in z).

Applying $\partial_w\partial_{\bar{z}}^3$ to the mapping equation and evaluating at $w = \bar{z} = \bar{w} = 0$, using $k(z,0) = 0$ and $\mu + 4i\nu = 0$, we find that

$$p_w\big|_{\Sigma} = 0.$$

Next, applying $\partial_w\partial_{\bar{z}}^j\partial_{\bar{w}}^{3-j}$, $j = 0, 1, 2$, to the mapping equation (4.2) and evaluating at $w = \bar{z} = \bar{w} = 0$ we find that

$$\begin{pmatrix} -4z & 0 \\ 2i & -2z \\ 0 & 3i \end{pmatrix} \cdot \begin{pmatrix} t_w \\ q_w \end{pmatrix} = \begin{pmatrix} il - 4i\bar{\nu}z^3\bar{\phi}^{(2,1)} - 16|\nu|^2z^2 - 4\alpha\nu z \\ -4i\bar{\nu}z^2\bar{f}^{(1,2)} - i\alpha z\bar{f}^{(1,2)} + im - 4i\bar{\nu}z^3\bar{\phi}^{(1,2)} + 4\bar{\nu}\phi_w \\ -5i\bar{\nu}z^2\bar{f}^{(0,3)} - i\alpha z\bar{f}^{(0,3)} + in + 18\bar{\nu}^3z^3 \end{pmatrix}.$$

Again, by the Kronecker-Capelli theorem, this overdetermined system is solvable iff

$$\Delta := \det\begin{pmatrix} -4z & 0 & il - 4i\bar{\nu}z^3\bar{\phi}^{(2,1)} - 16|\nu|^2z^2 - 4\alpha\nu z \\ 2i & -2z & -4i\bar{\nu}z^2\bar{f}^{(1,2)} - i\alpha z\bar{f}^{(1,2)} + im - 4i\bar{\nu}z^3\bar{\phi}^{(1,2)} + 4\bar{\nu}\phi_w \\ 0 & 3i & -5i\bar{\nu}z^2\bar{f}^{(0,3)} - i\alpha z\bar{f}^{(0,3)} + in + 18\bar{\nu}^3z^3 \end{pmatrix} = 0.$$

On the other hand, applying $\partial_w \partial_{\bar{z}}^j \partial_{\bar{w}}^{2-j}$, $j = 0, 1, 2$, and evaluating at $w = \bar{z} = \bar{w} = 0$, we obtain a system of 3 linear equations for r_w, s_w, and ϕ_w similarly to the first case. In fact,

$$\phi_w\big|_\Sigma = -16i\bar{\nu}^2 z^4 + \cdots$$

where the dots are the terms which are at most cubic in z. Plugging this into the determinant above and expanding it, the resulting polynomial is of degree at most 5. Collecting the term of degree 5 in z from the term $18\bar{\nu}^3 z^3$ of the $(3,3)$-entry and the term $4\bar{\nu}\phi_w$ in the $(2,3)$-entry, we have

$$\Delta = -96\bar{\nu}^3 z^5 + \cdots$$

where the dots represent terms of degree 4 or less in z. Thus, the vanishing of Δ implies that $\nu = 0$, as desired. $\quad\square$

Thus, from Lemmas 4.11 and 4.12 we have $\mu = -3i\nu$ and we use this from now on.

Lemma 4.13. *If $\lambda = 0$, then $\mu = -3i\nu$. Moreover,*

$$f_w\big|_\Sigma = \frac{5}{2}i\bar{\nu}z^2 + \frac{i\alpha z}{2}, \tag{4.40}$$

$$\phi_w\big|_\Sigma = \frac{1}{3}iz^2\left(2\bar{f}^{(1,2)} - 4f^{(1,2)} + \alpha^2\right) + \frac{1}{3}iz^3\left(18\alpha\bar{\nu} - 4i\bar{f}^{(0,3)}\right) + 6i\bar{\nu}^2 z^4 - 3i\nu z, \tag{4.41}$$

$$g_w\big|_\Sigma = 1, \tag{4.42}$$

and

$$f_{ww}\big|_\Sigma = 2\nu + f^{(1,2)}z + z^2\left(\frac{4}{3}i\bar{f}^{(0,3)} - \frac{13\alpha\bar{\nu}}{2}\right) - 15\bar{\nu}^2 z^3, \tag{4.43}$$

$$g_{ww}\big|_\Sigma = 4i\bar{\nu}z, \tag{4.44}$$

$$g_{www}\big|_\Sigma = \bar{g}^{(0,3)} + 2z\left(i\bar{f}^{(0,3)} - 3\alpha\bar{\nu}\right) - 36\bar{\nu}^2 z^2. \tag{4.45}$$

Proof. Since $g(z, w) = w(wr(z, w) + 1)$, we have $g_w\big|_\Sigma = 1$. From (4.7),

$$r(z, 0) = 2i\bar{\nu}z, \quad s(z, 0) = \frac{i}{2}\alpha z + \frac{1}{2}i\bar{\nu}z^2.$$

Therefore,

$$f_w\big|_\Sigma = zr + s\big|_\Sigma = \frac{5}{2}i\bar{\nu}z^2 + \frac{i\alpha z}{2},$$

as desired.

Next, we solve the following system for p, t and q along Σ:

$$
\begin{pmatrix} i & -4z & 0 \\ 0 & 2i & -2z \\ 0 & 0 & 3i \end{pmatrix} \cdot \begin{pmatrix} p \\ t \\ q \end{pmatrix} = \begin{pmatrix} 10i\nu z + \dfrac{4}{3}i(3\bar{f}^{(1,2)} - \bar{g}^{(0,3)})z^2 \\ 4\nu - 2\bar{f}^{(1,2)}z - \dfrac{4i}{3}\bar{f}^{(0,3)}z^2 + 4\bar{\nu}^2 z^3 \\ i\bar{g}^{(0,3)} - 2\bar{f}^{(0,3)}z - 6i\bar{\nu}^2 z^2 \end{pmatrix} .
$$

We find that

$$
\begin{pmatrix} p \\ t \\ q \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 6i & 12z & -8iz^2 \\ 0 & 3i & 2z \\ 0 & 0 & 2i \end{pmatrix} \cdot \begin{pmatrix} 10i\nu z + \dfrac{4}{3}i(3\bar{f}^{(1,2)} - \bar{g}^{(0,3)})z^2 \\ 4\nu - 2\bar{f}^{(1,2)}z - \dfrac{4i}{3}\bar{f}^{(0,3)}z^2 + 4\bar{\nu}^2 z^3 \\ i\bar{g}^{(0,3)} - 2\bar{f}^{(0,3)}z - 6i\bar{\nu}^2 z^2 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2\nu z \\ \dfrac{1}{3}iz\left(3\bar{f}^{(1,2)} + \bar{g}^{(0,3)}\right) - 2i\nu \\ -2\bar{\nu}^2 z^2 + \dfrac{1}{3}\bar{g}^{(0,3)} + \dfrac{2}{3}iz\bar{f}^{(0,3)} \end{pmatrix} .
$$

Applying the differential operators $\partial_z^j \partial_{\bar{w}}^{2-j} \partial_w$, $j = 0, 1, 2$, to the mapping equation (4.2) and evaluating along Σ, we obtain

$$
\begin{pmatrix} 0 & 4iz & 1 \\ -z & 1 & 0 \\ i & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} r_w \\ s_w \\ \phi_w \end{pmatrix} = \frac{i}{2} \begin{pmatrix} p \\ t \\ q \end{pmatrix} + \begin{pmatrix} -2\alpha z f_w + i\alpha\phi g_w \\ \dfrac{1}{2}i\alpha f_w - 3i\bar{\nu}z f_w - \dfrac{3}{2}\bar{\nu}\phi g_w \\ -2\bar{\nu}f_w \end{pmatrix} .
$$

The coefficient matrix on the left, denoted by D_3, is invertible with the inverse

$$
D_3^{-1} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & -iz \\ 1 & -4iz & -4z^2 \end{pmatrix}
$$

Using the formula for p, t, q and f_w, g_w along Σ, we find that

$$
r_w|_\Sigma = \frac{1}{6}\bar{g}^{(0,3)} + \frac{1}{6}z\left(-6\alpha\bar{\nu} + 2i\bar{f}^{(0,3)}\right) - 6\bar{\nu}^2 z^2, \tag{4.46}
$$

$$
s_w|_\Sigma = z\left(-\frac{1}{2}\bar{f}^{(1,2)} + \frac{1}{3}\bar{g}^{(0,3)} - \frac{\alpha^2}{4}\right) + z^2\left(-\frac{9\alpha\bar{\nu}}{4} + \frac{1}{3}i\bar{f}^{(0,3)}\right) + \nu - \frac{3}{2}\bar{\nu}^2 z^3, \tag{4.47}
$$

$$
\phi_w|_\Sigma = \frac{1}{3}iz^2\left(6\bar{f}^{(1,2)} - 4\bar{g}^{(0,3)} + 3\alpha^2\right) + \frac{1}{3}iz^3\left(18\alpha\bar{\nu} - 4i\bar{f}^{(0,3)}\right) + 6i\bar{\nu}^2 z^4 - 3i\nu z \tag{4.48}
$$

From the formula for ϕ_w, we can compute

$$
\phi^{(2,1)} = \frac{2}{3}i\left(6\bar{f}^{(1,2)} - 4\bar{g}^{(0,3)} + 3\alpha^2\right) .
$$

This and Lemma 4.10 imply the following "reflection identity"

$$6f^{(1,2)} + 6\bar{f}^{(1,2)} - 2g^{(0,3)} - 4\bar{g}^{(0,3)} + 3\alpha^2 = 0,$$

which, in turn, shows that $g^{(0,3)}$ is real and

$$g^{(0,3)} = \bar{g}^{(0,3)} = f^{(1,2)} + \bar{f}^{(1,2)} + \frac{\alpha^2}{2}.$$

Plugging this back into the formula for ϕ_w we obtain

$$\phi_w|_\Sigma = -3i\nu z + \frac{1}{3}iz^2\left(2\bar{f}^{(1,2)} - 4f^{(1,2)} + \alpha^2\right) + \frac{1}{3}iz^3\left(18\alpha\bar{\nu} - 4i\bar{f}^{(0,3)}\right) + 6i\bar{\nu}^2 z^4.$$

Since $f = z + w(zr + s)$, we have

$$f_w(z, 0) = zr_w(z, 0) + s_w(z, 0).$$

Plugging in the formulas for r_w and s_w along $w = 0$, we obtain the desired formula for $f_w(z, 0)$. The formulas for g_w and g_{ww} along $w = 0$ can be deduced similarly from the identity $g = w(1 + wr)$. The proof is complete. $\square$

Lemma 4.14. *If $\nu = 0$, then $f^{(0,3)} = 0$.*

Proof. Using the formula for f_w, ϕ_w, and g_w along Σ, we can produce another holomorphic equation for f, ϕ, and g by differentiating the mapping equation along the CR vector field. Indeed, we have

$$L\left(\varrho(H(z, \bar{w} + 2i|z|^2), \overline{H}(\bar{z}, \bar{w}))\right) = 0.$$

Substituting $\bar{w} = 0$ and using $\bar{f}(\bar{z}, 0) = \bar{z}, \bar{\phi}(\bar{z}, 0) = \alpha\bar{z}^2 + 2\nu\bar{z}^3, \bar{g}(\bar{z}, 0) = 0$, we find that

$$0 = 2iz + \Upsilon(z, 2i|z|^2)\left(2iz\bar{\phi}_{\bar{w}}(\bar{z}, 0) - \bar{\phi}_{\bar{z}}(\bar{z}, 0)\right) - f(z, 2i|z|^2)\left(4z\bar{f}_{\bar{w}}(\bar{z}, 0) + 2i\right)$$
$$-\phi(z, 2i|z|^2)\left(4|z|^2\bar{f}_{\bar{w}}(\bar{z}, 0) + 2iz\bar{\phi}(\bar{z}, 0) + 2i\bar{z}\right), \qquad (4.49)$$

where, as before, $\Upsilon := \phi g + if^2$. Substituting $\bar{z} = w/(2iz)$, we obtain an equation of the form

$$0 = 2iz + \Upsilon(z, w) + \cdots$$

Since $\nu = 0$, we have $\bar{\phi}_{\bar{w}}(\bar{z}, 0) = \frac{4}{3}f^{(0,3)}\bar{z}^3 +$ lower order terms. Clearing the denominator z^2, we obtain a holomorphic functional equation for $f(z, w), g(z, w)$, and $\phi(z, w)$. Setting $z = 0$, we obtain $0 = w^3 f^{(0,3)}\Upsilon(0, w)$. Thus,

$$f^{(0,3)}\Upsilon(0, w) = 0. \qquad (4.50)$$

On the other hand, with $\nu = 0$, setting $w = 0$ in the first holomorphic functional equation (4.31) and solving for $g(0, w)$, we find that

$$\phi(0, w) = \frac{\alpha f(0, w)^2}{-1 + i\alpha g(0, w)}.$$

Substituting $\phi(0, w)$ as above into (4.50), we find that

$$f^{(0,3)} f(0, w)^2 = 0.$$

Hence, either $f^{(0,3)} = 0$ or $f(0, w) = 0$. But the latter also implies that $f^{(0,3)} = 0$. The proof is complete. $\square$

Lemma 4.15. *If $\lambda = 0$, then $\sigma = 0$ and $\mu = -3i\nu$. Moreover,*

$$\phi^{(3,1)} = 12i\alpha\bar{\nu},$$

$$\phi^{(2,2)} = \frac{2i}{3}\left(g^{(0,4)} - 4f^{(1,3)}\right) - \frac{5\alpha}{3}f^{(1,2)} - \frac{8\alpha}{3}\bar{f}^{(1,2)} - \frac{1}{3}\alpha^3 - 12|\nu|^2,$$

$$\phi^{(1,3)} = 4i\nu\bar{f}^{(1,2)} - 2i\nu f^{(1,2)} - if^{(0,4)} + \frac{13}{2}i\alpha^2\nu,$$

$$\phi^{(0,4)} = 24\alpha\nu^2.$$

Proof. The solvability condition for the overdetermined system (4.39) for l, m, and n reads

$$\begin{pmatrix} -6z & 0 & 0 & 18i\nu\phi - z^2\bar{\phi}^{(3,1)} \\ 2i & -4z & 0 & -2z\bar{f}^{(2,2)} - z^2\bar{\phi}^{(2,2)} + (\alpha^2 - 4\bar{f}^{(1,2)} - 2i\bar{\phi}^{(2,1)})\phi \\ 0 & 3i & -2z & -2z\bar{f}^{(1,3)} + i\bar{g}^{(1,3)} - z^2\bar{\phi}^{(1,3)} + (6i\alpha\bar{\nu} - 3i\bar{\phi}^{(1,2)} - 2\bar{f}^{(0,3)})\phi \\ 0 & 0 & 4i & -2z\bar{f}^{(0,4)} + i\bar{g}^{(0,4)} - z^2\bar{\phi}^{(0,4)} \end{pmatrix} = 0.$$

$$(4.51)$$

Here we used $6\bar{\nu}^2 + i\bar{\phi}^{(0,3)} = 0$. Recall that $\phi(z, 0) = 2\bar{\nu}z^3 + \alpha z^2$ and $f(z, 0) = z$. Plugging these into the determinant and equating the coefficients of z^k, $k = 2, 3, 4, 5$, we obtain

$$\phi^{(3,1)} = -2i\left(3f^{(2,2)} - 2g^{(1,3)} + 9\alpha\bar{\nu}\right) = 4\left(2\bar{f}^{(0,3)} + 3i\alpha\bar{\nu}\right),$$

$$\phi^{(2,2)} = \alpha f^{(1,2)} + \frac{2}{3}i\left(g^{(0,4)} - 4f^{(1,3)}\right) - \frac{8}{3}\alpha g^{(0,3)} + \alpha^3 - 12|\nu|^2,$$

$$\phi^{(1,3)} = 2\alpha f^{(0,3)} - i\left(6\nu f^{(1,2)} + f^{(0,4)} - 4\nu g^{(0,3)}\right) - \frac{3}{2}i\alpha^2\nu,$$

$$\phi^{(0,4)} = -8i\nu f^{(0,3)}.$$

With these equations being satisfied, we can solve for l, m, and n along Σ. $\square$

Lemma 4.16. *If $\lambda = 0$, then*

$$\bar{g}^{(1,3)} = \frac{3}{4} \left(\bar{f}^{(2,2)} + 5\alpha\nu \right),$$
(4.52)

$$\bar{g}^{(0,4)} = \frac{1}{2}i \left(4\alpha f^{(1,2)} + 10\alpha \bar{f}^{(1,2)} - 4i\bar{f}^{(1,3)} - 4\alpha\bar{g}^{(0,3)} + 2\alpha^3 + 75|\nu|^2 \right),$$
(4.53)

$$\bar{f}^{(0,4)} = \frac{1}{2}\bar{\nu} \left(36f^{(1,2)} + 18\bar{f}^{(1,2)} - 16\bar{g}^{(0,3)} - 3\alpha^2 \right),$$
(4.54)

$$\bar{f}^{(0,3)} = -3i\alpha\bar{\nu}$$
(4.55)

Proof. Applying the fourth-order differential operators $\partial_w \partial_{\bar{z}}^j \partial_{\bar{w}}^{3-j}$, $j = 0, 1, 2, 3$, to the mapping equation (4.2) and evaluating at $w = \bar{z} = \bar{w} = 0$, we obtain a system of 4 linear equations of 3 unknowns that must be satisfied by p_w, t_w, and q_w along Σ. By the Kronecker-Capelli theorem again, the following determinant must vanish identically on the first Segre set:

$$\Delta := \det \begin{pmatrix} -z & 0 & 0 & -6i|\nu|^2 z^3 \\ i & -4z & 0 & il - 3i\bar{\nu}z^3 \bar{\phi}^{(2,1)} - 25|\nu|^2 z^2 - 5\alpha\nu z \\ 0 & 2i & -2z & -5i\bar{\nu}z^2 \bar{f}^{(1,2)} - i\alpha z \bar{f}^{(1,2)} + im - 3i\bar{\nu}z^3 \bar{\phi}^{(1,2)} + 2\bar{\nu}\phi_w \\ 0 & 0 & 3i & -5i\bar{\nu}z^2 \bar{f}^{(0,3)} - i\alpha z \bar{f}^{(0,3)} + in + 18\bar{\nu}^3 z^3 \end{pmatrix} = 0.$$

Expanding the determinant along the last column, the result is a polynomial of degree at most 6 in z. Collecting the terms of kind z^6, we easily see that they are canceled out. Since $n(z,0)$ is linear in z, the terms of degree 5 only come from the terms of kind z^2 in the $(4,4)$-entry and the terms of kind z^3 in the $(3,4)$-entry. Explicitly, equating the coefficient of z^5, we obtain

$$\bar{\nu} \left(2i\bar{f}^{(0,3)} - 3\bar{\phi}^{(1,2)} + 12\alpha\bar{\nu} \right) = 0.$$

Using the formula for $\phi^{(1,2)}$ in Lemma 4.10, we obtain

$$\bar{\nu} \left(3\alpha\bar{\nu} - i\bar{f}^{(0,3)} \right) = 0.$$

Thus,

$$i\bar{f}^{(0,3)} = 3\alpha\bar{\nu},$$

provided that $\nu \neq 0$. On the other hand, if $\nu = 0$, then this also holds by Lemma 4.14. Consequently, $g^{(1,3)} = 0$, $f^{(2,2)} = -5\alpha\bar{\nu}$, $\phi^{(3,1)} = 12i\alpha\bar{\nu}$. $\square$

Putting these calculations above together, we obtain

Lemma 4.17. *If $\lambda = 0$, then*

$$\phi_w\big|_\Sigma = -3i\nu z + \frac{1}{3}iz^2\left(2\bar{f}^{(1,2)} - 4f^{(1,2)} + \alpha^2\right) + 2i\alpha\bar\nu z^3 + 6i\bar\nu^2 z^4, \qquad (4.56)$$

$$f_{ww}\big|_\Sigma = 2\nu + f^{(1,2)}z - \frac{5}{2}\alpha\bar\nu z^2 - 15\bar\nu^2 z^3, \qquad (4.57)$$

$$g_{www}\big|_\Sigma = 2\,\mathrm{Re}\,f^{(1,2)} + \frac{\alpha^2}{2} - 36\bar\nu^2 z^2. \qquad (4.58)$$

When the equalities in Lemma 4.15 hold, the overdetermined system for l, m, n given in (4.39) can be solved uniquely. Explicitly,

$$l\big|_\Sigma = -\frac{1}{3}iz\left(2\bar{g}^{(1,3)} - 3\bar{f}^{(2,2)}\right) - 6i|\nu|^2 z^2 = -5i\alpha\nu z - 6i|\nu|^2 z^2, \qquad (4.59)$$

$$m\big|_\Sigma = \frac{1}{6}\bar\nu z^2\left(8f^{(1,2)} - 4\bar{f}^{(1,2)} + \alpha^2\right) - \frac{1}{6}iz\left(\bar{g}^{(0,4)} - 4\bar{f}^{(1,3)}\right) + \frac{1}{3}\bar{g}^{(1,3)}$$

$$= \frac{1}{6}\bar\nu z^2\left(8f^{(1,2)} - 4\bar{f}^{(1,2)} + \alpha^2\right) - \frac{1}{6}iz\left(\bar{g}^{(0,4)} - 4\bar{f}^{(1,3)}\right), \qquad (4.60)$$

$$n\big|_\Sigma = -2\bar\nu z^2 \bar{f}^{(0,3)} + \frac{1}{2}iz\bar{f}^{(0,4)} + \frac{1}{4}\bar{g}^{(0,4)}$$

$$= 6i\alpha\bar\nu^2 z^2 + \frac{1}{2}iz\bar{f}^{(0,4)} + \frac{1}{4}\bar{g}^{(0,4)}. \qquad (4.61)$$

Recall that $k\big|_\Sigma = 0$.

In the next step, we solve for p_w, t_w, and q_w along Σ. To this end, we apply $\partial_{\bar z}^j \partial_{\bar w}^{3-j} \partial_w$, $j = 0, 1, 2, 3$, to the mapping equation (4.2), evaluating along $\bar z = \bar w = w = 0$, and plugging in the formulas above for l, m, and n, to obtain

$$\begin{pmatrix} -z & 0 & 0 \\ i & -4z & 0 \\ 0 & 2i & -2z \\ 0 & 0 & 3i \end{pmatrix} \cdot \begin{pmatrix} p_w \\ t_w \\ q_w \end{pmatrix}$$

$$= \begin{pmatrix} -6i|\nu|^2 z^3 \\ il + 2\bar\nu z^3(4\bar{f}^{(1,2)} - 2f^{(1,2)} - \alpha^2) - 25|\nu|^2 z^2 - 5\alpha\nu z \\ im + 2\bar\nu z^3(2\bar{f}^{(0,3)} - 3i\alpha\bar\nu) - 5i\bar\nu z^2 \bar{f}^{(1,2)} - i\alpha z \bar{f}^{(1,2)} + 2\bar\nu\phi_w \\ in - 5i\bar\nu z^2 \bar{f}^{(0,3)} - i\alpha z \bar{f}^{(0,3)} + 18\bar\nu^3 z^3 \end{pmatrix}$$

$$= \begin{pmatrix} -6i|\nu|^2 z^3 \\ 2\bar\nu z^3(4\bar{f}^{(1,2)} - 2f^{(1,2)} - \alpha^2) - 19|\nu|^2 z^2 \\ im + 18i\alpha\bar\nu^2 z^3 - 5i\bar\nu z^2 \bar{f}^{(1,2)} - i\alpha z \bar{f}^{(1,2)} + 2\bar\nu\phi_w \\ -\frac{i}{4}\bar{g}^{(0,4)} - \left(\frac{1}{2}\bar{f}^{(0,4)} + 3\alpha^2\bar\nu\right)z - 21\alpha\bar\nu^2 z^2 + 18\bar\nu^3 z^3 \end{pmatrix}$$

when $w = 0$. Here we have used various formulas for the third-order derivatives of ϕ at $z = w = 0$ obtained earlier in Lemma 4.10. The resulting overdetermined system

is solvable if and only if the corresponding augmented matrix is degenerate. By direct computation the determinant is equal to

$$2z^4 \left(2\bar{\nu}\left(10f^{(1,2)} + \bar{f}^{(1,2)}\right) - 2\bar{f}^{(0,4)} - 11\alpha^2\bar{\nu}\right)$$
$$+ 2z^3 \left(6\alpha\bar{f}^{(1,2)} + 2i\left(\bar{g}^{(0,4)} - 2\bar{f}^{(1,3)}\right) + 75|\nu|^2\right).$$

Equating the coefficients of powers of z to be zero, we have

$$\bar{g}^{(0,4)} = 3i\alpha\bar{f}^{(1,2)} + 2\bar{f}^{(1,3)} + \frac{75i}{2}|\nu|^2, \tag{4.62}$$

$$\bar{f}^{(0,4)} = \bar{\nu}f^{(1,2)} + 10\bar{\nu}\bar{f}^{(1,2)} - \frac{11}{2}\bar{\nu}\alpha^2. \tag{4.63}$$

On the other hand, substituting $g^{(0,3)}$ and $g^{(0,4)}$ into Lemma 4.15 we find that

$$\phi^{(2,2)} = \frac{1}{3}\left(10\alpha f^{(1,2)} - 8\alpha\bar{f}^{(1,2)} - 4if^{(1,3)} - \alpha^3 + 39|\nu|^2\right). \tag{4.64}$$

A "reflection identity" for $\phi^{(2,2)}$ is used in the following lemma.

Lemma 4.18. *Assume that $\lambda = 0$, then*

$$3\alpha\left(f^{(1,2)} + \bar{f}^{(1,2)}\right) + 2i\left(f^{(1,3)} - \bar{f}^{(1,3)}\right) + 123|\nu|^2 = 0. \tag{4.65}$$

Moreover

$$\phi_{ww}\big|_\Sigma = 6\alpha\nu z + \frac{1}{6}(285|\nu|^2 - \alpha^3 + 16\alpha f^{(1,2)} - 2\alpha\bar{f}^{(1,2)} - 4i\bar{f}^{(1,3)})z^2$$
$$- \frac{1}{3}\bar{\nu}(5\alpha^2 - 68f^{(1,2)} + 46\bar{f}^{(1,2)})z^3 + \frac{25}{2}\alpha\bar{\nu}^2 z^4 - 9\bar{\nu}^3 z^5.$$

Proof. With all the identities above, we can solve for p_w, t_w and q_w. Precisely

$$p_w = 6i|\nu|^2 z^2,$$

$$t_w = \frac{1}{2}\bar{\nu}z^2\left(2f^{(1,2)} - 4\bar{f}^{(1,2)} + \alpha^2\right) + \frac{13|\nu|^2 z}{4},$$

$$q_w = -6i\bar{\nu}^3 z^3 + 7i\alpha\bar{\nu}^2 z^2 + \frac{1}{12}i\bar{\nu}z\left(20f^{(1,2)} + 2\bar{f}^{(1,2)} + \alpha^2\right) -$$
$$\frac{1}{24}i\left(-6\alpha\bar{f}^{(1,2)} + 4i\bar{f}^{(1,3)} - 75|\nu|^2\right).$$

Applying $\partial_w^2 \partial_{\bar{w}}^j \partial_{\bar{z}}^{2-j}$, $j = 0, 1, 2$, we obtain a system of the form:

$$\begin{pmatrix} 0 & 4iz & 1 \\ -z & 1 & 0 \\ i & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} r_{ww} \\ s_{ww} \\ \phi_{ww} \end{pmatrix} = i\begin{pmatrix} p_w \\ t_w \\ q_w \end{pmatrix} + \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix},$$

where

$$
\begin{aligned}
V_1 &= -2\alpha z f_{ww} + \alpha z g_{ww}(4s - i\alpha z) + 2i\alpha \phi_w + \frac{25}{2}\alpha \bar{\nu}^2 z^4 + 5\alpha^2 \bar{\nu} z^3 + \frac{\alpha^3 z^2}{2}, \\
V_2 &= \frac{1}{2} i\alpha f_{ww} - 3i\bar{\nu} z f_{ww} - \frac{3}{2} \bar{\nu} g_{ww}\phi - 3\bar{\nu}\phi_w + \frac{75}{4} i\bar{\nu}^3 z^4 + \frac{15}{2} i\alpha \bar{\nu}^2 z^3 + \frac{3}{4} i\alpha^2 \bar{\nu} z^2, \\
V_3 &= -2\bar{\nu} f_{ww}.
\end{aligned}
$$

Solving for ϕ_{ww} we find that

$$
\phi^{(2,2)} = \frac{1}{3}\left(16\alpha f^{(1,2)} - 2\alpha \bar{f}^{(1,2)} - 4i\bar{f}^{(1,3)} - \alpha^3 + 285|\nu|^2\right).
$$

Compare this with formula (4.64) for $\phi^{(2,2)}$ as above, we are done.   □

Lemma 4.19. *If $\lambda = 0$, then $\nu = \mu = \sigma = 0$. The components f, ϕ, and g of the map satisfy the following three holomorphic functional equations:*

$$
-4wzf + iw^2\phi + 4z^2 g + \alpha w^2 \Upsilon = 0. \tag{4.66}
$$

$$
6z(\alpha w - 2i)f - w\phi + 12iz^2 + \left(6i\alpha w - w^2\left(2f^{(1,2)} - 4\bar{f}^{(1,2)} + \alpha^2\right)\right)\Upsilon = 0, \tag{4.67}
$$

$$
A_0(z,w)f(z,w) + B_0(z,w)\phi(z,w) + C_0(z,w)\Upsilon(z,w) = 0, \tag{4.68}
$$

for

$$
A_0(z,w) = 24wz\bar{f}^{(1,2)} + 24i\alpha z, \tag{4.69}
$$

$$
B_0(z,w) = 4iw^2 \bar{f}^{(1,2)} - 8iw^2 f^{(1,2)} - i\alpha^2 w^2 - 12i, \tag{4.70}
$$

$$
\begin{aligned}
C_0(z,w) = {}& w^2\left(2\alpha f^{(1,2)} - 4if^{(1,3)} - 16\alpha \bar{f}^{(1,2)} + \alpha^3\right) \\
& - 8iw\left(2f^{(1,2)} - 16\bar{f}^{(1,2)} + \alpha^2\right) - 12\alpha.
\end{aligned} \tag{4.71}
$$

Proof. Using the formula for f_w, ϕ_w, and g_w along Σ, we can produce another holomorphic functional equation for f, ϕ, and g by differentiating the mapping equation along the CR vector field. Indeed, we have

$$
L\left(L\left(\varrho(H(z, \bar{w} + 2i|z|^2), \overline{H}(\bar{z}, \bar{w}))\right)\right)\Big|_{\bar{w}=0} = 0.
$$

Expanding the derivatives and substituting the formulas above, we have

$$
0 = A(z,w)f(z,w) + B(z,w)\phi(z,w) + C(z,w)\Upsilon(z,w) - 192\nu w z^4, \tag{4.72}
$$

where, by direct calculation,

$$
\begin{aligned}
A(z, w) = {} & 96wz^4 \bar{f}^{(1,2)} + 360\nu^2 w^3 z^2 + 120i\alpha\nu w^2 z^3 \\
& + 480\nu w z^3 + 384i\bar{\nu}z^5 + 96i\alpha z^4,
\end{aligned}
\tag{4.73}
$$

$$
\begin{aligned}
B(z, w) = {} & 16iw^2 z^3 \bar{f}^{(1,2)} - 32iw^2 z^3 f^{(1,2)} - 135i\nu^2 w^4 z + 24\alpha\nu w^3 z^2 \\
& - 4i\alpha^2 w^2 z^3 - 216i\nu w^2 z^2 - 96\bar{\nu}wz^4 - 48iz^3,
\end{aligned}
\tag{4.74}
$$

$$
\begin{aligned}
C(z, w) = {} & 4w^2 z^3 \left(2\alpha f^{(1,2)} - 4if^{(1,3)} - 16\alpha \bar{f}^{(1,2)} + \alpha^3 - 285|\nu|^2 \right) \\
& - 4iw^3 z^2 \left(46\nu f^{(1,2)} - 68\nu \bar{f}^{(1,2)} + 5\alpha^2 \nu \right) \\
& - 16iwz^3 \left(4f^{(1,2)} - 32\bar{f}^{(1,2)} + 2\alpha^2 \right) - 48\alpha z^3 - 144\alpha\nu w^2 z^2 + 144i\nu wz^2 \\
& + 75\alpha\nu^2 w^4 z + 288i\nu^2 w^3 z + 27i\nu^3 w^5 - 288i\alpha\bar{\nu}wz^4 - 288\bar{\nu}z^4.
\end{aligned}
\tag{4.75}
$$

On the other hand, from (4.31), we have

$$
g(z, w) = \frac{w\left(4iz^2 f(z, w) + wf(z, w)^2 (\alpha z - i\nu w) + wz\phi(z, w) \right)}{w^2 \phi(z, w)(\nu w + i\alpha z) + 4iz^3}.
$$

Substituting this into Υ, we have

$$
\Upsilon(z, w) = \frac{-z(2zf(z, w) - iw\phi(z, w))^2}{\nu w^3 \phi(z, w) + i\alpha w^2 z\phi(z, w) + 4iz^3}.
$$

Plugging this into (4.72), canceling the nonzero term $z/(\nu w^3 \phi(z, w) + i\alpha w^2 z\phi(z, w) + 4iz^3)$, and setting $z = 0$, we obtain

$$
108i\nu^3 w^7 \phi(0, w)^2 = 0.
$$

(In this step, the only term of degree 1 in z in $B(z, w)$ and the term containing no z in $C(z, w)$ are important.) Thus, either $\nu = 0$ or $\phi(0, w) = 0$. But in the latter case, we have $0 = \phi^{(0,3)} = -6i\nu^2$ and hence $\nu = 0$, as desired.

Plugging these into (4.31) and canceling z we obtain (4.66). Similarly, we get (4.67) from Lemma 4.14 and the third equation (4.68) from (4.72). The proof is complete. $\qquad\square$

From now on, we use the fact that $\nu = \mu = \sigma = 0$. Then, collecting from above,

$$
H(z, 0) = (z, \alpha z^2, 0),
$$

$$
H_w(z, 0) = \left(\frac{i\alpha z}{2}, \frac{1}{3}iz^2 \left(2\bar{f}^{(1,2)} - 4f^{(1,2)} + \alpha^2 \right), 1 \right),
$$

$$
H_{ww}(z, 0) = \left(f^{(1,2)} z, -\frac{1}{6} \left(\alpha^3 - 16\alpha f^{(1,2)} + 2\alpha \bar{f}^{(1,2)} + 4i\bar{f}^{(1,3)} \right) z^2, 0 \right).
$$

For each triple of parameters $(\alpha, f^{(1,2)}, f^{(1,3)}) \in \mathbb{R} \times \mathbb{C} \times \mathbb{C}$ satisfying (4.65) with $\nu = 0$, the system of holomorphic functional equations (4.66), (4.67), and (4.68) above

has a unique algebraic solution, which may have a singularity at the origin. Even in the case the solution is holomorphic near the origin, the resulting map does not necessarily send $\mathbb{H}^3$ into $\mathcal{X}$, see Example 5.3 below. In the next step, we determine all the triples of parameters $\left(\alpha, f^{(1,2)}, f^{(1,3)}\right)$ such that the solution of this system is a genuine map sending $\mathbb{H}^3$ into $\mathcal{X}$. We write $f^{(1,2)} = \eta + i\xi$, with $\eta, \xi \in \mathbb{R}$. From Lemma 4.18, we can write $f^{(1,3)} = \gamma + (3i/2)\alpha\eta$ with $\gamma \in \mathbb{R}$. The mapping equation (4.32) gives the following constrains on these parameters.

Lemma 4.20. *Under the assumptions of Lemma 4.19 the following three equations hold:*

$$\alpha^5 - 4\alpha^3\eta + 4\alpha\left(90\xi^2 + \eta^2\right) - 72\gamma\xi = 0, \tag{4.76}$$

$$23\alpha^4\xi - 4\alpha^3\gamma - 56\alpha^2\xi\eta + 8\alpha\gamma\eta - 1116\xi^3 + 20\xi\eta^2 = 0, \tag{4.77}$$

$$7\alpha^6 - 240\alpha^4\eta + 2304\alpha^2\left(142\xi^2 + \eta^2\right) - 290304\alpha\gamma\xi + 62208\gamma^2 - 4096\left(333\xi^2\eta + \eta^3\right) = 0. \tag{4.78}$$

The following will be useful for the proof of Lemma 4.20.

Lemma 4.21. *If $\lambda = 0$, then*

$$f(0, w) = \phi(0, w) = 0, \quad \phi_z(0, w) = 0.$$

Proof. Setting $z = 0$ in the first holomorphic functional equation (4.66), and solving for $\phi(0, w)$ we have

$$\phi(0, w) = -\frac{\alpha f(0, w)^2}{1 - i\alpha g(0, w)}.$$

Setting $z = 0$ into the second holomorphic functional equation (4.67) and plugging in $\phi(0, w)$ from above we have

$$w^2 f(0, w)^2 \left(2f^{(1,2)} - 4\bar{f}^{(1,2)} + \alpha^2\right) = 0.$$

Assume, by contradiction, that $f(0, w) \neq 0$, then $f^{(1,2)} = \alpha^2/2$.

Setting $z = 0$ in the third holomorphic functional equation (4.68) and plugging in $\phi(0, w)$ from above we have

$$-2w^2 f(0, w)^2 \left(5\alpha f^{(1,2)} - 2if^{(1,3)} - 10\alpha\bar{f}^{(1,2)} + \alpha^3\right) = 0.$$

Solving for $f^{(1,3)}$ we have

$$f^{(1,3)} = \frac{3i\alpha^3}{4}.$$

Plugging these two values into the holomorphic functional equations, we obtain precisely the map which satisfies $f(0, w) = 0$. Thus, in any case, $f(0, w) = 0$ and hence $\phi(0, w) = 0$. The proof is complete. □

For the proof of Lemma 4.20 we associate weight 1 to z and weight 2 to w and consider a weighted homogeneous expansion of H, i.e.,

$$f(z, w) = z + \sum_{i \geqslant 2} f_i(z, w), \quad \phi(z, w) = \sum_{j \geqslant 2} \phi_j(z, w), \quad g(z, w) = w + \sum_{k \geqslant 3} g_k(z, w),$$

where $h_m(z, w)$ is a weighted homogeneous polynomial of order $m \geqslant 2$. Collecting weighted homogeneous terms of weight $k + 1$ for $k \geqslant 3$ in the mapping equation gives:

$$\operatorname{Re}\big(ig_{k+1}(z, w) + 2\bar{z}f_k(z, w) + \bar{z}^2\phi_{k-1}(z, w)\big) + \cdots = 0,$$

for $(z, w) \in \mathbb{H}^3$. The dots $\cdots$ denote terms, which involve parts of the weighted homogeneous expansion of the map of lower order, which appear for $k \geqslant 5$.

The lower weight components are computed above as follows:

$$H_1 = (z, 0, 0) \qquad\qquad H_5 = \left(\tfrac{1}{2}f^{(1,2)}zw^2, 0, 0\right)$$
$$H_2 = \left(0, \alpha z^2, w\right) \qquad\qquad H_6 = \left(0, \tfrac{1}{4}\phi^{(2,2)}z^2w^2, \tfrac{1}{6}g^{(0,3)}w^3\right)$$
$$H_3 = \left(\tfrac{i\alpha}{2}zw, 0, 0\right) \qquad\qquad H_7 = \left(\tfrac{1}{6}f^{(1,3)}zw^3, 0, 0\right),$$
$$H_4 = \left(0, \tfrac{1}{2}\phi^{(2,1)}z^2w, 0\right)$$

where $\phi^{(2,1)} = \tfrac{2i}{3}\left(2\bar{f}^{(1,2)} - 4f^{(1,2)} + \alpha^2\right)$, $\phi^{(2,2)} = \tfrac{1}{3}\left(16\alpha f^{(1,2)} - 2\alpha\bar{f}^{(1,2)} - 4i\bar{f}^{(1,3)} - \alpha^3\right)$, and $g^{(0,3)} = \alpha^2/2 + 2\operatorname{Re}f^{(1,2)}$.

Lemma 4.22. *For every $j \geqslant 2$, $g_{2j+1} = f_{2j} = \phi_{2j-1} = 0$ and $g_{2j} = g_{2j,j}w^j$ with $g_{2j,j}$ being real.*

Proof. Observe that the conclusion holds for $j = 0, 1, 2, 3$. We argue by induction as follows. Setting $z = 0$ in the mapping equation of weight $2j+1$, noticing that $g_{2j+1}(0, \bar{w}) = 0$ (as $\bar{w}$ has weight 2), and using Lemma 4.21 above ($f(0, w) = \phi(0, w) = 0$), we immediately conclude that g_{2j+1} must vanish. Plugging this back into the mapping equation of weight $2j + 1$, the remaining terms are divisible by z. Moreover, f_{2j} is divisible by z^2 while $f_{2j-2} = 0$ by the induction assumption. Dividing the equation by z and setting $z = 0$, we obtain $f_{2j} = 0$. Plugging these two into the mapping equation and arguing similarly, we find that $\phi_{2j-1} = 0$. □

In particular,

$$g_8 = \frac{1}{24}g^{(0,4)}w^4 = \frac{1}{24}\left(2f^{(1,3)} - 3i\alpha f^{(1,2)}\right)w^4.$$

In the sequel we use the following notation: Expanding terms h_k of weighted order k in z and w we name the coefficients according to the powers of w and write:

$$h_k(z, w) = h_{k,0}z^k + h_{k,1}z^{k-2}w + \cdots + h_{k,\lfloor k/2 \rfloor}z^{k-2\lfloor k/2 \rfloor}w^{\lfloor k/2 \rfloor}.$$

Comparing the terms of weight 10 in the mapping equation, we have

$$
0 = ig_{10,5}\left(w^5 - \bar{w}^5\right) + 2\sum_{j=0}^{4} f_{2j+1}\bar{f}_{9-2j} + 2\operatorname{Im}\sum_{l=1}^{3} \phi_{2l}\sum_{j=1}^{4-l}\bar{\phi}_{10-2l-2j}g_{2j}
$$

$$
+ \operatorname{Re}\sum_{l=1}^{4}\bar{\phi}_{2l}\sum_{j=0}^{4-l} f_{2j+1}f_{9-2l-2j}\Bigg|_{w=\bar{w}+2iz\bar{z}}.
$$

Observe that in each monomial appearing in the right-hand side, the highest degree in $\bar{w}$ is 4. Collecting the terms $z\bar{z}\bar{w}^4$ appearing in g_{10}, $f_1\bar{f}_9$, $\bar{f}_1 f_9$, and $f_5\bar{f}_5$, we have

$$2\operatorname{Re} f_{9,4} - 5g_{10,5} + \alpha\operatorname{Im} f_{7,3} + |f_{5,2}|^2 = 0.$$

Collecting the terms of kind $z\bar{z}^3\bar{w}^3$ appearing only in $f_1\bar{f}_9$ we find that $f_{9,3} = 0$. Thus, f_9 is monomial, that is $f_9 = f_{9,4}zw^4$. Collecting terms of kind $z^2\bar{z}^2\bar{w}^3$ we have

$$48f_{9,4} - 120g_{10,5} - 3i\operatorname{Re}\phi_{8,3} - 10\alpha^2\operatorname{Im} f_{5,2}$$

$$+ 24|f_{5,2}|^2 - 16i\operatorname{Im} f_{5,2}^2 - 6i\alpha(3f_{7,3} - \bar{f}_{7,3}) = 0.$$

The coefficient of $z^5\bar{z}^5$ gives:

$$4f_{9,4} - i\phi_{8,3} - 4g_{10,5} - 2i\alpha^2\operatorname{Re} f_{5,2} = 0.$$

The coefficient of $z^4\bar{z}^4\bar{w}$ gives:

$$576f_{9,4} - 108i\phi_{8,3} - 720g_{10,5} - \alpha^4 - 128|f_{5,2}|^2 - 72i\alpha f_{7,3}$$

$$- 104\alpha^2 f_{5,2} + 100\alpha^2\bar{f}_{5,2} + 32f_{5,2}^2 + 128\bar{f}_{5,2}^2 = 0.$$

And finally, the coefficient of $z^3\bar{z}^3\bar{w}^2$ gives:

$$864f_{9,4} - 108i\phi_{8,3} - 1440g_{10,5} - \alpha^4 - 16|f_{5,2}|^2 - 216i\alpha f_{7,3}$$

$$- 92\alpha^2 f_{5,2} + 88\alpha^2\bar{f}_{5,2} - 64f_{5,2}^2 + 224\bar{f}_{5,2}^2 = 0.$$

Solving for $(f_{9,4}, \phi_{8,3}, g_{10,5})$ and using (4.18) in the above system of five equations we obtain:

$$f_{9,4} = \frac{1}{144}\left(5\alpha^4 + 928|f_{5,2}|^2 + 72i\alpha f_{7,3} + 4\alpha^2 f_{5,2} + 16\alpha^2 \bar{f}_{5,2} - 352f_{5,2}^2 - 448\bar{f}_{5,2}^2\right),$$

$$\phi_{8,3} = -\frac{i}{12}\left(\alpha^4 + 176|f_{5,2}|^2 + 24i\alpha f_{7,3} - 12\alpha^2 f_{5,2} + 16\alpha^2 \bar{f}5, 2 - 64f_{5,2}^2 - 96\bar{f}_{5,2}^2\right),$$

$$g_{10,5} = \frac{1}{72}\left(\alpha^4 + 200|f_{5,2}|^2 + 4\alpha^2 \operatorname{Re} f_{5,2} - 160\operatorname{Re} f_{5,2}^2\right).$$

Considering terms of weight 12 in the mapping equation, we obtain from $z^8\bar{z}^4$ that $f_{11,3} = 0$, from $z^2\bar{z}^4\bar{w}^3$ that $\phi_{10,3} = 0$ and from $z^5\bar{z}^3\bar{w}^2$ that $f_{11,4} = 0$, which shows that $f_{11} = f_{11,5}z^2w^5$ and $\phi_{10} = \phi_{10,4}z^2w^4$ are monomials.

Furthermore, the coefficients of $z, \bar{z}$ and $\bar{w}$ allow to solve for the triple $(f_{11,5}, \phi_{10,4}, g_{12,6})$ and provide an additional equation for $(\alpha, f_{5,2}, f_{7,3})$. More precisely, the coefficient of $z\bar{z}w^5$ gives

$$24\operatorname{Re} f_{11,5} - 72g_{12,6} - \alpha^3 \operatorname{Im} f_{5,2} + 6\alpha^2 \operatorname{Re} f_{7,3} + 8\alpha \operatorname{Im} f_{5,2}^2 + 24\operatorname{Re}(f_{7,3}\bar{f}_{5,2}) = 0.$$

The coefficient of $z^5\bar{z}^5\bar{w}$ gives

$$160if_{11,5} - 192ig_{12,6} + 32\phi_{10,4} + \alpha^5 + 20\alpha^3 f_{5,2} - 24\alpha^3 \bar{f}_{5,2} - 32\alpha f_{5,2}^2$$
$$+ 64\alpha|f_{5,2}|^2 - 8i\alpha^2 f_{7,3} = 0.$$

The coefficient of $z^3\bar{z}^3\bar{w}^3$ gives

$$720if_{11,5} + 72\phi_{10,4} - 1440ig_{12,6} + 8\alpha^5 + 1312\alpha|f_{5,2}|^2 + 43\alpha^3 f_{5,2} - 29\alpha^3 \bar{f}_{5,2} - 496\alpha f_{5,2}^2$$
$$- 568\alpha \bar{f}_{5,2}^2 + 264i\bar{f}_{5,2}f_{7,3} + 24if_{5,2}\bar{f}_{7,3} + 192\operatorname{Im}(f_{5,2}f_{7,3}) + 66i\alpha^2 f_{7,3} + 6i\alpha^2 \bar{f}_{7,3} = 0.$$

The coefficient of $z^6\bar{z}^6$ gives

$$288if_{11,5} + 72\phi_{10,4} - 288ig_{12,6} + \alpha^5 - 160\alpha|f_{5,2}|^2 + 50\alpha^3 f_{5,2} - 64\alpha^3 \bar{f}_{5,2} + 40\alpha f_{5,2}^2$$
$$+ 160\alpha \bar{f}_{5,2}^2 - 36i\alpha^2 f_{7,3} = 0.$$

Solving the above system for $(f_{11,5}, \bar{f}_{11,5}, \phi_{10,4}, g_{12,6})$ we obtain

$$f_{11,5} = \frac{i}{48}(\alpha^5 + 320\alpha|f_{5,2}|^2 - 11\alpha^3 f_{5,2} + 15\alpha^3 \bar{f}_{5,2} - 120\alpha f_{5,2}^2 - 184\alpha \bar{f}_{5,2}^2 + 14i\alpha^2 f_{7,3}$$
$$- 2i\alpha^2 \bar{f}_{7,3} - 8if_{5,2}\bar{f}_{7,3} - 88i\bar{f}_{5,2}f_{7,3} - 64\operatorname{Im}(f_{5,2}f_{7,3})),$$

$$\bar{f}_{11,5} = -\frac{i}{96}(\alpha^5 - 64\alpha|f_{5,2}|^2 + 28\alpha^3 f_{5,2} - 24\alpha^3 \bar{f}_{5,2} + 128\alpha \bar{f}_{5,2}^2 - 8i\alpha^2 f_{7,3} - 16i\alpha^2 \bar{f}_{7,3}$$
$$- 64if_{5,2}\bar{f}_{7,3} + 256i\bar{f}_{5,2}f_{7,3} + 256\operatorname{Im}(f_{5,2}f_{7,3})),$$

$$\phi_{10,4} = \frac{1}{48}(3\alpha^5 + 1152\alpha|f_{5,2}|^2 - 62\alpha^3 f_{5,2} + 86\alpha^3 \bar{f}_{5,2} - 416\alpha f_{5,2}^2 - 688\alpha \bar{f}_{5,2}^2 + 76i\alpha^2 f_{7,3}$$
$$- 4i\alpha^2 \bar{f}_{7,3} - 16if_{5,2}\bar{f}_{7,3} - 176i\bar{f}_{5,2}f_{7,3} - 128\operatorname{Im}(f_{5,2}f_{7,3})),$$

$$g_{12,6} = \frac{i}{576}(\alpha^5 + 704\alpha|f_{5,2}|^2 - 46\alpha^3 f_{5,2} + 50\alpha^3 \bar{f}_{5,2} - 272\alpha f_{5,2}^2 - 464\alpha \bar{f}_{5,2}^2$$
$$- 24\alpha^2 \operatorname{Im} f_{7,3} - 48i f_{5,2} \bar{f}_{7,3} - 528i \bar{f}_{5,2} f_{7,3} - 384 \operatorname{Im}(f_{5,2} f_{7,3})).$$

The remaining equation (or equivalently the difference of the conjugate of the formula of $f_{11,5}$ and $\bar{f}_{11,5}$ above) using (4.18) is given by (4.76).

Considering terms of weight 14, we obtain from $z^9 \bar{z}^3 \bar{w}$ that $f_{13,3} = 0$ and $z^2 \bar{z}^6 \bar{w}^3$ gives $\phi_{12,3} = 0$. The coefficient of $z^9 \bar{z}^5$ implies $f_{13,4} = 0$, the coefficient of $z^3 \bar{z} \bar{w}^5$ shows $f_{13,5} = 0$ and the coefficient of $z^5 \bar{z}^3 \bar{w}^3$ implies $\phi_{12,4} = 0$. Hence, $f_{13} = f_{13,6} z^2 w^6$ and $\phi_{12} = \phi_{12,5} z^4 w^5$ are monomials.

Furthermore, the coefficient of $z \bar{z} \bar{w}^6$ gives

$$1152 \operatorname{Re} f_{13,6} - 4032 g_{14,7} + 9\alpha^6 + 1760\alpha^2 |f_{5,2}|^2 + 76\alpha^4 \operatorname{Re} f_{5,2} - 1312\alpha^2 \operatorname{Re} f_{5,2}^2$$
$$- 3584 \operatorname{Re} f_{5,2}^3 + 4608|f_{5,2}|^2 \operatorname{Re} \bar{f}_{5,2} + 576|f_{7,3}|^2 - 120\alpha^3 \operatorname{Im} f_{7,3} + 384\alpha \operatorname{Im}(f_{5,2} f_{7,3})$$
$$+ 1056\alpha \operatorname{Im}(f_{5,2} \bar{f}_{7,3}) = 0,$$

the coefficient of $z^7 \bar{z}^7$ gives

$$576 f_{13,6} - 144i\phi_{12,5} - 576 g_{14,7} + \alpha^6 - 1072\alpha^2|f_{5,2}|^2 + 92\alpha^4 f_{5,2} - 136\alpha^4 \bar{f}_{5,2} + 352\alpha^2 f_{5,2}^2$$
$$+ 736\alpha^2 \bar{f}_{5,2}^2 - 72i\alpha^3 f_{7,3} = 0,$$

the coefficient of $z^3 \bar{z}^3 \bar{w}^4$ gives

$$4320 f_{13,6} - 360i\phi_{12,5} - 10080 g_{14,7} + 31\alpha^6 + 7312\alpha^2|f_{5,2}|^2 - 154\alpha^4 f_{5,2} + 246\alpha^4 \bar{f}_{5,2}$$
$$- 2576\alpha^2 f_{5,2}^2 - 4056\alpha^2 \bar{f}_{5,2}^2 + 768 f_{5,2}^3 - 5792 \bar{f}_{5,2}^3 + 15296 f_{5,2} \bar{f}_{5,2}^2 - 8384 f_{5,2}^2 \bar{f}_{5,2}$$
$$+ 252i\alpha^3 f_{7,3} - 72i\alpha^3 \bar{f}_{7,3} + 864|f_{7,3}|^2 - 720i\alpha f_{5,2} \bar{f}_{7,3} - 2160i\alpha \bar{f}_{5,2} f_{7,3}$$
$$- 1728\alpha \operatorname{Im}(f_{5,2} f_{7,3}) = 0,$$

and the coefficient of $z^4 \bar{z}^4 \bar{w}^3$ gives

$$1440 f_{13,6} - 180i\phi_{12,5} - 2520 g_{14,7} + 8\alpha^6 + 1236\alpha^2|f_{5,2}|^2 + 11\alpha^4 f_{5,2} - 20\alpha^4 \bar{f}_{5,2}$$
$$- 470\alpha^2 f_{5,2}^2 - 646\alpha^2 \bar{f}_{5,2}^2 + 400 f_{5,2}^3 - 672 \bar{f}_{5,2}^3 + 2416 f_{5,2} \bar{f}_{5,2}^2 - 1840 f_{5,2}^2 \bar{f}_{5,2}$$
$$+ 24i\alpha^3 f_{7,3} - 18i\alpha^3 \bar{f}_{7,3} + 72|f_{7,3}|^2 - 624i\alpha \bar{f}_{5,2} f_{7,3} - 144i\alpha f_{5,2} \bar{f}_{7,3} + 204i\alpha f_{5,2} f_{7,3}$$
$$- 180i\alpha \bar{f}_{5,2} \bar{f}_{7,3} = 0.$$

We then solve the above system of four equations for the quadruple $(f_{13,6}, \bar{f}_{13,6}, \phi_{12,5}, g_{14,7})$ to obtain

$$f_{13,6} = \frac{1}{5760}(-39\alpha^6 + 4\alpha^4(288 f_{5,2} - 335 \bar{f}_{5,2}) - 16\alpha^2(1483|f_{5,2}|^2 - 570 f_{5,2}^2 - 1013 \bar{f}_{5,2}^2)$$

$$+ 64(|f_{5,2}|^2(70f_{5,2} + 578\bar{f}_{5,2}) - 180f_{5,2}^3 - 401\bar{f}_{5,2}^3) - 144i\alpha^3(8f_{7,3} - \bar{f}_{7,3})$$
$$+ 5184|f_{7,3}|^2 - 288i\alpha(f_{5,2}(4f_{7,3} + \bar{f}_{7,3}) - 29\bar{f}_{5,2}f_{7,3})),$$

$$\bar{f}_{13,6} = \frac{1}{5760}(-9\alpha^6 + \alpha^4(568\bar{f}_{5,2} - 636f_{5,2}) - 16\alpha^2(163|f_{5,2}|^2 - 162f_{5,2}^2 - 41\bar{f}_{5,2}^2)$$
$$- 64(|f_{5,2}|^2(878f_{5,2} - 1526\bar{f}_{5,2}) - 96f_{5,2}^3 + 677\bar{f}_{5,2}^3) - 24i\alpha^3(5f_{7,3} - 19\bar{f}_{7,3})$$
$$+ 5184|f_{7,3}|^2 + 192i\alpha(f_{5,2}(23f_{7,3} + 8\bar{f}_{7,3}) - \bar{f}_{5,2}(22f_{7,3} + 31\bar{f}_{7,3}))),$$

$$\phi_{12,5} = \frac{i}{1440}(35\alpha^6 - 4\alpha^4(486f_{5,2} - 661\bar{f}_{5,2}) + 16\alpha^2(2075|f_{5,2}|^2 - 744f_{5,2}^2 - 1381\bar{f}_{5,2}^2)$$
$$- 64(|f_{5,2}|^2(134f_{5,2} + 226\bar{f}_{5,2}) - 128f_{5,2}^3 - 207\bar{f}_{5,2}^3) + 48i\alpha^3(37f_{7,3} - 3\bar{f}_{7,3})$$
$$- 2880|f_{7,3}|^2 + 96i\alpha(f_{5,2}(14f_{7,3} - 3\bar{f}_{7,3}) - \bar{f}_{5,2}(73f_{7,3} + 6\bar{f}_{7,3}))),$$

$$g_{14,7} = \frac{1}{2880}(3\alpha^6 + 4\alpha^4(16f_{5,2} - 7\bar{f}_{5,2}) - 16\alpha^2(39f|_{5,2}|^2 - 23f_{5,2}^2 - 46\bar{f}_{5,2}^2)$$
$$- 64(|f_{5,2}|^2(32f_{5,2} - 176\bar{f}_{5,2}) + 26f_{5,2}^3 + 97\bar{f}_{5,2}^3) - 48i\alpha^3 f_{7,3} + 1152|f_{7,3}|^2$$
$$+ 96i\alpha(f_{5,2}(f_{7,3} - 3\bar{f}_{7,3}) + \bar{f}_{5,2}(7f_{7,3} - 3\bar{f}_{7,3}))).$$

Using the above expressions and (4.18), the remaining equations reduce to (4.77) and (4.78). This finishes the proof of Lemma 4.20.

Lemma 4.23. *The solutions of* (4.76)–(4.78) *are given as follows: If* $\alpha \neq 0$, *then the only solution is* $\xi = \gamma = 0$ *and* $\eta = \alpha^2/2$. *If* $\alpha = 0$, *then a solution satisfies* $\xi = 0$ *and* $6\gamma^2 - \eta^3 = 0$.

Proof. If $\alpha = 0$, then (4.76) shows that $\gamma\xi = 0$. When $\gamma = 0$, we have $\eta = \xi = 0$. If $\xi = 0$, then (4.77) reduces to $6\gamma^2 - \eta^3 = 0$.

Consider the case $\alpha \neq 0$. Assume $\xi \neq 0$, then (4.76) shows

$$\gamma = \frac{1}{72\xi}\alpha(1 + 360\xi^2 - 4\eta(1 - \eta)).$$

The remaining two equations then become

$$18\xi^2(3 - 4\eta(4 - 5\eta)) - (1 - 2\eta)^3 - 20088\xi^4 = 0, \tag{4.79}$$
$$(1 - 2\eta)^4 - 36\xi^2(1 - 2\eta)^2(1 + 6\eta) + 2592(53 - 111\eta)\xi^4 = 0. \tag{4.80}$$

Note that $\eta = 53/111$ does not lead to a solution. So we can assume $\eta \neq 53/111$. Combining (4.79) and (4.80), to eliminate ξ^4, we obtain

$$(1 - 2\eta)((1 - 2\eta)^2(181 - 382\eta) - 36\xi^2(287 - 2\eta(925 - 1296\eta))) = 0. \tag{4.81}$$

Observe that $\eta = 1/2$ leads to $\xi = 0$, which is not allowed. Also, if the coefficient of ξ^2 is zero, this gives a contradiction to the fact that $\xi \in \mathbb{R}$. Solving (4.81) for ξ^2 gives

$$\xi^2 = \frac{(1-2\eta)^2(182-382\eta)}{36(287-2\eta(925-1296\eta))}. \tag{4.82}$$

Substituting this into the equations (4.79) and (4.80) implies that $\eta \in \{53/111, 1/2\}$, which we already excluded, or $\eta = 51/22$. Assuming the latter leads to a negative expression in the right-hand side of (4.82). Thus the case $\xi \neq 0$ does not lead to any solution. Assuming $\xi = 0$ directly leads to $\gamma = 0$ and $\eta = \alpha^2/2$, which completes the proof. $\square$

When $\alpha \neq 0$, the 4-jet of the map at 0 is completely fixed, while for $\alpha = 0$ there is still one parameter to be determined. In the latter case we need to consider weight 16 in the weighted homogeneous expansion of the mapping equation.

Assuming $\alpha = 0$, we obtain from the coefficient of $z^{12}\bar{z}^4$ that $f_{15,3} = 0$, from $z^7\bar{z}w^4$ that $f_{15,4} = 0$, from $z^{11}\bar{z}^5$ that $\phi_{14,3} = 0$, from $z^5\bar{z}w^5$ that $f_{15,5} = 0$, from $z^9\bar{z}^5\bar{w}$ that $\phi_{14,4} = 0$, from $z^3\bar{z}w^6$ that $f_{15,6} = 0$, and from $z^7\bar{z}^5w^2$ that $\phi_{14,5} = 0$. This shows that $f_{15} = f_{15,7}z^2w^7$ and $\phi_{14} = \phi_{14,6}z^4w^6$ are both monomials.

Furthermore, the coefficient of $z\bar{z}\bar{w}^7$ gives

$$36\,\mathrm{Re}\,f_{15,7} - 144g_{16,8} - 8f_{5,2}^2(10f_{7,3} + 7\bar{f}_{7,3}) - 11\bar{f}_{5,2}^2(f_{7,3} + 4\bar{f}_{7,3})$$
$$+ |f_{5,2}|^2(152f_{7,3} + 143\bar{f}_{7,3}) = 0,$$

the coefficient of $z^7\bar{z}^7\bar{w}$ gives

$$14f_{15,7} - 3i\phi_{14,6} - 16g_{16,8} = 0,$$

the coefficient of $z^5\bar{z}^5\bar{w}^3$ gives

$$630f_{15,7} - 90i\phi_{14,6} - 1008g_{16,8} + 32f_{5,2}^2(f_{7,3} - \bar{f}_{7,3}) + \bar{f}_{5,2}^2(209f_{7,3} + 100\bar{f}_{7,3})$$
$$+ |f_{5,2}|^2(23\bar{f}_{7,3} - 164f_{7,3}) = 0,$$

and the coefficient of $z^8\bar{z}^8$ gives

$$4if_{15,7} + \phi_{14,6} - 4ig_{16,8} = 0.$$

Solving the above system for $(f_{15,7}, \bar{f}_{15,7}, \phi_{14,6}, g_{16,8})$, we obtain

$$f_{15,7} = \frac{1}{54}\left(32f_{5,2}^2(f_{7,3} - \bar{f}_{7,3}) + \bar{f}_{5,2}^2(209f_{7,3} + 100\bar{f}_{7,3}) + |f_{5,2}|^2(23\bar{f}_{7,3} - 164f_{7,3})\right),$$

$$\bar{f}_{15,7} = \frac{2}{9}\left(f_{5,2}^2(28f_{7,3} + 6\bar{f}_{7,3}) + \bar{f}_{5,2}^2(55f_{7,3} + 36\bar{f}_{7,3}) - |f_{5,2}|^2(79f_{7,3} + 30\bar{f}_{7,3})\right),$$

$$\phi_{14,6} = -\frac{i}{27}\left(32f_{5,2}^2(f_{7,3} - \bar{f}_{7,3}) + \bar{f}_{5,2}^2(209f_{7,3} + 100\bar{f}_{7,3}) + |f_{5,2}|^2(23\bar{f}_{7,3} - 164f_{7,3})\right),$$

$$g_{16,8} = \frac{1}{108}\left(32f_{5,2}^2(f_{7,3} - \bar{f}_{7,3}) + \bar{f}_{5,2}^2(209f_{7,3} + 100\bar{f}_{7,3}) + |f_{5,2}|^2(23\bar{f}_{7,3} - 164f_{7,3})\right).$$

When using $\xi = 0$, the remaining coefficients reduce to the condition $\gamma\eta^2 = 0$, which, together with Lemma 4.23, implies that $f^{(1,2)} = f^{(1,3)} = 0$. This fixes the 4-jet of the map at 0 in the case $\alpha = 0$.

Note that we have the following expressions:

$$H(z,0) = (z, \alpha z^2, 0), \quad H_w(z,0) = \left(\frac{i\alpha}{2}z, 0, 1\right), \quad H_{ww}(z,0) = \left(\frac{\alpha^2}{2}z, \frac{\alpha^3}{2}z^2, 0\right).$$

Using the information about the 4-jet of H in Lemma 4.19 we obtain the following holomorphic functional equations for H:

$$4iz(wf - zg) + \alpha w^2 f^2 + w^2(1 - i\alpha g)\phi = 0$$
$$2iz^2 - (2i - \alpha w)zf - \alpha w f^2 - w(1 - i\alpha g)\phi = 0$$
$$4i\alpha z^2 f - z(2i - \alpha w)(\alpha f^2 + (1 - i\alpha g)\phi) = 0.$$

Solving this system of equations we get the family of maps, when writing $\alpha = 2\beta$,

$$r_\beta(z,w) = \frac{1}{1 - \beta^2 w^2} \left(z(1 + i\beta w), 2\beta z^2, w\right).$$

By the partial normal form in Theorem 3.1 we can assume $\beta \in \{-1, 0, 1\}$. Taking $\beta \in \{-1, 1\}$ gives r_1 and r_{-1} in Theorem 1.2 and r_0 corresponds to ℓ. We finish Case 2 when $\lambda = 0$. Together with Case 1 treated in the previous section, this completes the proof of part (a) of Theorem 1.2, giving a complete classification of CR transversal maps from $\mathbb{H}^3$ into $\mathcal{X}$.

The pairwise inequivalences of these 4 maps are clear from their partial normal forms.

4.3. Nowhere CR transversal maps

In this section we discuss nowhere CR transversal maps. This proves part (b) of Theorem 1.2. Indeed, if a CR map is CR transversal at one point, it is CR transversal at any point in its domain, as can be checked for the maps listed in part (a) of Theorem 1.2, see also Remark 2.

Lemma 4.24. *Any smooth CR map H from $\mathbb{H}^3$ into $\mathcal{X}$, which is not CR transversal at any point of $\mathbb{H}^3$ is equivalent to a map $(z,w) \mapsto (0, \phi(z,w), 0)$ for a CR function ϕ satisfying $\phi(0,0) = 0$.*

Note that in [47, Example 2.5], Xiao–Yuan also gave examples of nowhere CR transversal maps from the sphere into the boundary of the classical domain of type IV.

Proof. By homogeneity we can assume that H sends 0 to 0. If H is nowhere CR transversal in $\mathbb{H}^3$ it satisfies the equation:

$$(g(z,w) - \bar{g}(\bar{z}, \bar{w}))(1 - |\phi(z,w)|^2) - 2i|f(z,w)|^2 - 2i \operatorname{Re}\left(f(z,w)^2 \bar{\phi}(\bar{z}, \bar{w})\right) = 0, \quad (4.83)$$

for all $(z, w, \bar{z}, \bar{w})$ in a neighborhood of 0. Setting $\bar{z} = \bar{w} = 0$ shows $g = 0$. Note that $\phi = 0$ implies $f = 0$. Assume that $\phi \neq 0$. We write $f = \sum_{\ell \geqslant \ell_0} f_\ell$ and $\phi = \sum_{k \geqslant k_0} \phi_k$, where f_r and ϕ_r are weighted homogeneous polynomials of order r w.r.t. the weight $(1, 2)$ for (z, w) and $\ell_0, k_0 \geqslant 0$. Collecting terms of weight m in (4.83) we obtain:

$$2 \sum_{i=0}^{m} f_i \bar{f}_{m-i} + \sum_{j=0}^{m} \sum_{k=0}^{j} \bar{f}_{m-j} \bar{f}_{j-k} \phi_k + \sum_{l=0}^{m} \sum_{n=0}^{l} f_{m-l} f_{l-n} \bar{\phi}_n = 0, \qquad (4.84)$$

where we skip the arguments. We assume $\phi_{k_0} \neq 0$ and $\phi_k = 0$ for $k < k_0$. Considering $m = k_0$, shows $\sum_{i=0}^{k_0} f_i \bar{f}_{k_0 - i} = 0$, which implies $f_r = 0$ for $r \leqslant k_0$. Considering $m = 2k_0 + 2$ shows $f_{k_0+1} = 0$. Inductively, assuming $f_{k_0+s} = 0$ for $s < r$ and considering $m = 2(k_0 + r)$ for $r \geqslant 2$, we obtain that $f_{k_0+r} = 0$. This shows $f = 0$ and completes the proof. $\square$

4.4. Proof of Corollary 1.3

We can transform the maps given in Theorem 1.2, part (a), into proper holomorphic maps from $\mathbb{B}^2$ into D_3^{IV} using the birational transformation Φ defined in (2.5). First, let γ_0 be the automorphism of the ball given by

$$\gamma_0(z,w) = \left(\frac{2\sqrt{2}z}{3-w}, \frac{3w-1}{3-w}\right)$$

and $\mathcal{C}$ be the Cayley transform

$$\mathcal{C}(z,w) = \left(\frac{z}{1+w}, \; i\left(\frac{1-w}{1+w}\right)\right)$$

which sends $\mathbb{S}^3 \setminus \{(0, -1)\}$ onto the Heisenberg hypersurface $\mathbb{H}^3$. Then we obtain the map R_0 as in (i) via

$$R_0 = \Phi \circ \ell \circ \mathcal{C} \circ \gamma_0.$$

Moreover,

$$\varrho_{D_3^{\mathrm{IV}}} \circ R_0 = \varrho_{\mathbb{S}^3},$$

which means that R_0 has vanishing Ahlfors tensor: $\mathcal{A}(R_0) = 0$.

The family R_β obtained by composing Φ with r_β and the Cayley transform, i.e.,

$$R_\beta = \Phi \circ r_\beta \circ \mathcal{C} \circ \gamma_0$$

gives a 1-parameter smooth deformation of R_0 by proper holomorphic maps. This family has the same properties as the family r_β does, that all R_β's with $\beta > 0$ are mutually equivalent, and a similar statement holds for the case $\beta < 0$, yet R_0 is not equivalent to R_β for $\beta \neq 0$. This shows that, in contrast to the higher dimensional, but small codimensional case, treated in [47], R_0 is *not* rigid. Observe that R_β is rational of degree two with a rather simple formula. For each $\beta \neq 0$, R_β is equivalent to a polynomial (quadratic) map.

Clearly, the map $P_1(z,w) = (zw, (z^2 - w^2)/2, i(z^2 + w^2)/2)$ is a proper polynomial map sending $\mathbb{B}^2$ into D_3^{IV}. On the boundary, P_1 sends $\mathbb{S}^3$ into the smooth boundary part $\mathcal{R}$ transversally. Since

$$\varrho_{D_3^{\mathrm{IV}}} \circ P_1 = (1 + |z|^2 + |w|^2)\, \varrho_{\mathbb{S}^3},$$

we obtain, with $Z_1 = \bar{w}\partial_z - \bar{z}\partial_w \in \Gamma(T^{(1,0)}\mathbb{S}^3)$, that the Ahlfors invariant of P_1 is

$$\mathcal{A}(P_1)(Z_1, \overline{Z}_1) = \frac{1}{2}.$$

Similarly, the map $P_{-1}(z,w) = (z, w^2/2, iw^2/2)$ satisfies

$$\varrho_{D_3^{\mathrm{IV}}} \circ P_{-1} = (1 - |z|^2 + |w|^2)\, \varrho_{\mathbb{S}^3}$$

while

$$\tilde{\rho}_{D_3^{\mathrm{IV}}} \circ P_{-1}\bigg|_{\mathbb{S}^3} = -1 + |z|^2 + \frac{|w|^4}{2} \leqslant 0,$$

with the equality appearing if and only if $w = 0$. Thus, P_{-1} sends the circle $\{(e^{it}, 0)\} \subset \mathbb{S}^3$ into the singular locus of $\partial D_3^{\mathrm{IV}}$ and sends $\mathbb{S}^3 \setminus \{(e^{it}, 0)\}$ into the smooth boundary part $\mathcal{R} \subset \partial D_3^{\mathrm{IV}}$ transversally. The Ahlfors invariant of P_{-1} is defined on $\mathbb{S}^3 \setminus \{(e^{it}, 0)\}$:

$$\mathcal{A}(P_{-1})(Z_1, \overline{Z}_1) = -\frac{1}{2|w|^2}, \quad w \neq 0.$$

Thus, these three maps R_0, P_1, and P_{-1} are mutually inequivalent. Moreover, $R_\alpha \sim P_1$ for $\alpha > 0$ and similarly $R_\alpha \sim P_{-1}$ for $\alpha < 0$ (in contrast, the trace of the Ahlfors tensor of CR maps into the sphere is always nonnegative [34]).

The map I satisfies

$$\varrho_{D_3^{\mathrm{IV}}} \circ I = \varrho_{\mathbb{S}^3}$$

and thus has vanishing Ahlfors tensor. Consequently, I is of geometric rank zero at all points. Clearly, any smooth deformation of I along the sphere must be equivalent to I.

None of the rational maps R_0, P_1, and P_{-1} can be equivalent to the irrational map I, since all the automorphisms of the source and target are rational. Hence, these four maps

are pairwise inequivalent. This can also be verified by analyzing elementary geometric properties of the maps as already explained in Remark 2.

Finally, assume $H\colon \mathbb{B}^2 \to D_3^{\mathrm{IV}}$ is a proper holomorphic map that extends smoothly to a boundary point $p \in \mathbb{S}^3$. We can assume that $p = (0,1)$ and $H(p) = \left(0, \frac{1}{2}, \frac{i}{2}\right)$. Then $\tilde{H} := \Phi^{-1} \circ H \circ \mathcal{C}^{-1}$ defines a germ at the origin of CR maps sending $\mathbb{H}^3$ into $\mathcal{X}$. Thus, it must belong to one of the five equivalence classes of the germs represented by r_0, r_1, r_{-1}, ι, and t for some nowhere CR transversal map t as given in Lemma 4.24. But if $\tilde{H}$ is equivalent to t, then H cannot be proper as can be easily checked. Hence, $\tilde{H}$ must be in one of the equivalence classes represented by the germs of CR transversal maps, which are the four equivalence classes of the germs of $\Phi^{-1} \circ F \circ \mathcal{C}^{-1}$ with $F \in \{R_0, P_1, P_{-1}, I\}$. In fact, the four germs of CR maps $\Phi^{-1} \circ F \circ \mathcal{C}^{-1}$, $F \in \{R_0, P_1, P_{-1}, I\}$ are pairwise inequivalent as can be easily checked using the Ahlfors invariant and the irrationality. Thus, there are local CR automorphisms $\psi \in \mathrm{Aut}(\mathcal{X}, 0)$ and $\gamma \in \mathrm{Aut}(\mathbb{H}^3, 0)$ such that $\psi \circ \tilde{H} \circ \gamma^{-1} = \Phi^{-1} \circ F \circ \mathcal{C}^{-1}$ near the origin for some $F \in \{R_0, P_1, P_{-1}, I\}$. Thus, if $\tilde{\psi} := \Phi \circ \psi \circ \Phi^{-1}$ and $\tilde{\gamma} := \mathcal{C}^{-1} \circ \gamma \circ \mathcal{C}$, then, as germs at p, $\tilde{\psi} \circ H \circ \tilde{\gamma}^{-1} = F \in \{R_0, P_1, P_{-1}, I\}$. But $\tilde{\psi}$ is a global automorphism of D_3^{IV} by Theorem 2.3 and $\tilde{\gamma}$ is a global automorphism of the unit ball. This completes the proof of Corollary 1.3.

Remark 6. Although P_1 and P_{-1} have quite simple formulas, to the best of the authors' knowledge, they didn't appear before in the literature. Take the family of rational maps $r_\beta \colon \mathbb{H}^3 \to \mathcal{X}$ as above and transform it into a family of rational maps $\Phi \circ r_\beta \circ \mathcal{C}$ to get various representatives of the equivalence classes of $P_{\pm 1}$. For certain values of β, the formulas simplify. For example, if $\beta = -1/4$, then the map $\Phi \circ r_\beta \circ \mathcal{C}$ sends $\mathbb{B}^2$ into $D_3^{\mathrm{IV}} \cap \{z_3 - iz_2 = 0\}$. This suggests that we substitute $z_3 = iz_2$ into $\varrho_{D_3^{\mathrm{IV}}}$ to obtain

$$\varrho_{D_3^{\mathrm{IV}}}(z_1, z_2, iz_2) = 1 - 2|z_1|^2 - 4|z_2|^2 + |z_1|^4 = (1 - |z_1|^2)^2 - 4|z_2|^2.$$

From this, we can easily find the desired map P_{-1}. The formula for P_1 can be constructed easily by analyzing the map $\Phi \circ r_\beta \circ \mathcal{C}$ with $\beta = 1/4$.

5. Examples

Observe that $\mathcal{X} \cap \{\zeta = 0\}$ is CR equivalent to the Heisenberg hypersurface $v = |z|^2$. In fact, $\mathcal{X} \cap \{\zeta = 0\}$ is the image of $\mathbb{H}^3$ under the linear map $\ell(z, w)$ in part (i) of Theorem 1.2. The formula for the stability group of $\mathrm{Aut}(\mathcal{X}, 0)$ suggests we consider the variety $V = \{w\zeta + iz^2 = 0\} \subset \mathbb{C}^3$ which has a singularity at the origin. On V, we have $\mathrm{Re}(z^2\bar{\zeta}) = -|\zeta|^2 v$ and thus on $\mathcal{X} \cap V$,

$$v = \frac{|z|^2 - v|\zeta|^2}{1 - |\zeta|^2},$$

implying $v = |z|^2$. The real subvariety $V \cap \mathcal{X}$ contains a complex variety $(0, \zeta, 0)$, and a graph over the Heisenberg hypersurface $v = |z|^2$ given by $(z, -iz^2/w, w)$, $w \neq 0$. All

germs of nontransversal CR maps at the origin send an open set in $\mathbb{H}^3$ into this complex variety, see Lemma 4.24. From Lemma 2.2, we find that $V \cap \mathcal{X}$ is "invariant" in the following sense: For $(\tilde{z}, \tilde{\zeta}, \tilde{w})$ in the stability group of the form (2.1), we have

$$\tilde{w}\tilde{\zeta} + i\tilde{z}^2 = -\lambda^2 u^2 (w\zeta + iz^2)/\delta.$$

That is $\gamma(\mathcal{X} \cap V) \subset \mathcal{X} \cap V$ for all $\gamma \in \mathrm{Aut}(\mathcal{X}, 0)$. From the observation above, we can construct examples of maps from $\mathbb{H}^3$ into $\mathcal{X}$ as follows.

Example 5.1. Each map in the family of rational maps $H_t(z, w) = (z, -iz^2/(w+t), w+t)$, $t \in \mathbb{R}$, sends $\mathbb{H}^3 \setminus \{(0, -t)\}$ into $\mathcal{X} \cap V$ transversally, but does not extend holomorphically to the point $(0, -t)$. It turns out that at any point $p \in \mathbb{H}^3 \setminus \{(0, -t)\}$, H_t represents a germ at p that is equivalent to the germ of ℓ at the origin. In fact,

$$\tilde{\varrho}_{\mathcal{X}} \circ H_t = |w + t + 2i|^{-2} \varrho_{\mathbb{H}^3},$$

and hence outside the pole of H_t, the Ahlfors tensor $\mathcal{A}(H_t)$ vanishes identically. Thus, H_t must be equivalent to either ℓ or ι. But the latter possibility is ruled out by the rationality and H_t is equivalent to ℓ. Transferring this map into a map from the unit ball into the type IV model, we obtain a map that is equivalent to R_0. Explicitly, with $t = 0$, we have

$$\Phi \circ H_0 \circ \mathcal{C} \circ \gamma_1 = -R_0,$$

for

$$\gamma_1(z, w) = \left(-\frac{2\sqrt{2}z}{w+3}, -\frac{3w+1}{w+3} \right)$$

is an automorphism of $\mathbb{B}^2$ exchanging the origin and $(0, -1/3)$.

Example 5.2. As an application of the transitive part of the automorphisms in section 2.2, we explicitly verify that the rational map $R : \mathbb{B}^2 \to D_3^{\mathrm{IV}}$ from [47, Theorem 1.4], given by

$$R(z_1, z_2) = \left(z_1, \frac{z_1^2/2 - z_2^2 + z_2}{\sqrt{2}(1 - z_2)}, \frac{i(z_1^2/2 + z_2^2 - z_2)}{\sqrt{2}(1 - z_2)} \right),$$

corresponds to the linear map $\ell : \mathbb{H}^3 \to \mathcal{X}, \ell(z, w) = (z, 0, w)$ from Theorem 1.2. Note that R sends $p = (0, 1/2) \in \mathbb{B}^2$ to $q = (0, 1/2\sqrt{2}, -i/2\sqrt{2}) \in D_3^{\mathrm{IV}}$. We define $\Psi_1 : \mathbb{C}^3 \setminus \{z_2 - iz_3 = \sqrt{2}\} \to \mathbb{C}^3$, given by

$$\Psi_1(z_1, z_2, z_3) = \left(\frac{-2z_1}{\sqrt{2} - z_2 + iz_3}, \frac{2z_2 + 2iz_3 - \sqrt{2}(z_1^2 + z_2^2 + z_3^2)}{2(\sqrt{2} - z_2 + iz_3)}, \frac{2(\sqrt{2}i + iz_2 + z_3)}{\sqrt{2} - z_2 + iz_3} \right),$$

which locally sends $\mathcal{R}$ into $\mathcal{X}$ and q to $(0, 1/2, 2i) \in \mathcal{X}$, and $\Psi_2 : \mathbb{C}^2 \setminus \{w = -i\} \to \mathbb{C}^2$, given by

$$\Psi_2(z, w) = \left(\frac{i - w}{i + w}, \frac{2iz}{i + w} \right),$$

which sends $\mathbb{H}^3$ into $\mathbb{S}^3$ and $(1/2, i)$ to p. We consider

$$
\begin{aligned}
\hat{R}(z, w) =& \Psi_1 \circ R \circ \Psi_2 \\
=& \left(\frac{2\sqrt{2}(1 - 2z - 2izw + w^2)}{-1 + 4z + (6i - 4iz + w)w}, \frac{1 - 4z + 2iw(1 + 2z) + 8z^2 - w^2}{-1 + 4z + (6i - 4iz + w)w}, \right. \\
& \left. 2i + \frac{4i(i - w)^2}{-1 + 4z + (6i - 4iz + w)w} \right),
\end{aligned}
$$

which locally maps $\mathbb{H}^3$ to $\mathcal{X}$ and $\hat{p} = (0, 1)$ to $\hat{q} = \frac{1}{3}(-2\sqrt{2}i, 1, 2i)$. Next, we compose $\hat{R}$ with automorphisms to ensure that it sends 0 to 0. More precisely, we define

$$\varphi(z, w) = (z, w + 1),$$

which is an automorphism of $\mathbb{H}^3$, and

$$\varphi_1'(z, \zeta, w) = \left(\frac{2\sqrt{2}z}{3 - \zeta}, \frac{3\zeta - 1}{3 - \zeta}, w - \frac{iz^2}{3 - \zeta} \right),$$

$$\varphi_2'(z, \zeta, w) = (z + i(1 + \zeta), \zeta, 2z + w + i(1 + \zeta)),$$

which originate from choosing $\lambda' = 1/2$ in (2.4) and $b = i$ and $r = 0$ in (2.3) respectively. If we set

$$
\begin{aligned}
\check{R}(z, w) =& \varphi_2' \circ \varphi_1' \circ \hat{R} \circ \varphi \\
=& \left(\frac{2(2i(z - (1 - i))z - 2izw + w^2)}{-2((i - 1) + z)^2 + ((2 + 4i) - 4iz)w + w^2}, \right. \\
& \frac{2z(3z - (2 - 2i)) - w(2 - 4iz) - w^2}{-2((i - 1) + z)^2 + ((2 + 4i) - 4iz)w + w^2}, \\
& \left. \frac{(4 + 4i)w((1 + i) - 2iz + w)}{-2((i - 1) + z)^2 + ((2 + 4i) - 4iz)w + w^2} \right),
\end{aligned}
$$

we obtain a map, which, again, locally sends $\mathbb{H}^3$ into $\mathcal{X}$ and 0 to 0. For the final step, we define the following automorphisms of $\mathbb{H}^3$ and $\mathcal{X}$ respectively, fixing 0,

$$\phi(z, w) = \left(\frac{2z + w}{(2 + 2i)(i + z) + w}, \frac{(i - 1)w}{(2 + 2i)(i + z) + w} \right),$$

$$\phi'(z, \zeta, w) = \left(-2i + \frac{8 + 4iz}{i(z - 2i)^2 + (\zeta - 1)w}, \frac{(4 + iz)z - 4i\zeta + (\zeta - 1)w}{i(z - 2i)^2 + (\zeta - 1)w}, \right.$$

$$\left. \frac{-4iw}{i(z - 2i)^2 + (\zeta - 1)w} \right),$$

which are obtained by following the procedure in the proof of Theorem 3.1. It can be verified that the map $L = \phi' \circ \check{R} \circ \phi$ agrees with ℓ.

Remark 7. Carrying out the analogous steps as in Example 5.2 it can be shown that the irrational map $I : \mathbb{B}^2 \to D_3^{\mathrm{IV}}$ given by $I(z, w) = \left(z, w, 1 - \sqrt{1 - z^2 - w^2} \right) / \sqrt{2}$ found by Xiao–Yuan in [47, Theorem 1.4] corresponds to the irrational map ι obtained in Theorem 1.2. More precisely, it is possible to obtain a map $\check{I}$ from I, which sends $(\mathbb{H}^3, 0)$ to $(\mathcal{X}, 0)$. After that, bringing $\check{I}$ to the partial normal form given in Theorem 3.1, it can be verified that the 4-jets at the origin of this map and ι agree. Since the formulas involved are more complicated than in the rational case we refrain from displaying them here.

Example 5.3. The mapping equation (4.2) can be regarded as a system of infinitely many linear and nonlinear equations for the Taylor coefficients of the components of the map under consideration. Each solution to this system gives rise to a holomorphic or formal CR map sending the germ $(\mathbb{H}^3, 0)$ into $(\mathcal{X}, 0)$ as a subset or formally. We give an example of a holomorphic map whose Taylor coefficients solve all *but* three equations of weight 14 as well as three holomorphic functional equations in Lemma 4.19, yet the map does *not* send $\mathbb{H}^3$ into $\mathcal{X}$. Indeed, consider the following family of holomorphic maps

$$H := (f, \phi, g) = \left(z + 2tzw^3, -4itz^2w^2, w + tw^4 \right), \; t \in \mathbb{R},$$

which satisfies the functional equations in Lemma 4.19. If $\Sigma_1 \colon (z, \bar{z}, \bar{w}) \mapsto (z, \bar{w} + 2iz\bar{z})$ is a parametrization of the (complexification) of the Heisenberg hypersurface, then

$$\widetilde{\varrho} \circ H_t \circ \Sigma_1 = -4t^2 \bar{w}^3 z \bar{z} (\bar{w} + 2iz\bar{z})^3.$$

Hence, H_t sends $\mathbb{H}^3$ into $\mathcal{X}$ if and only if $t = 0$.

Notes added after acceptance of article

Since the first revision of this paper was submitted, the authors were able to construct a new family of CR maps from the Heisenberg hypersurface $\mathbb{H}^5 \subset \mathbb{C}^3$ into the real hypersurface $\mathcal{X}_m \subset \mathbb{C}^m$, $m \geqslant 4$, where $\mathcal{X}_m$ is defined for each $m = 3, 4, \ldots$ as follows.

$$\mathcal{X}_m := \left\{ (Z, \zeta, w) \in \mathbb{C}^{m-2} \times \mathbb{C} \times \mathbb{C} : (1 - |\zeta|^2) \operatorname{Im} w - Z\overline{Z}^t - \operatorname{Re}\left(\bar{\zeta} Z Z^t \right) = 0, \; |\zeta| < 1 \right\}.$$

The new family is very similar to the family r_β constructed in Section 4.2 for the case $m = 3$. Precisely, the maps

$$\tilde{r}_\beta(z_1, z_2, w) := \frac{1}{1 - \beta^2 w^2} \left(z_1(1 + i\beta w), z_2(1 - i\beta w), 2\beta(z_1^2 - z_2^2), w \right), \ \beta \in \mathbb{R},$$

send the germ $(\mathbb{H}^5, 0)$ into the germ $(\mathcal{X}_4, 0)$. By computing the Ahlfors invariant of $\tilde{r}_\beta$, we can easily see that $\tilde{r}_\beta$ is not equivalent to $\tilde{r}_0$ for $\beta \neq 0$.

As in the proof of Corollary 1.3, from $\tilde{r}_\beta$ we obtain proper holomorphic maps $\tilde{R}_\beta \colon \mathbb{B}^3 \subset \mathbb{C}^3 \to D_4^{\mathrm{IV}} \subset \mathbb{C}^4$ such that $\tilde{R}_0$ is the rational isometry with respect to the canonical Bergman metrics while $\tilde{R}_\beta$ is *not* equivalent to $\tilde{R}_0$ if $\beta \neq 0$. Explicitly, the map

$$P(z_1, z_2, w) = \left(z_1, z_2 w, \frac{z_2^2 - w^2}{2}, \frac{i(z_2^2 + w^2)}{2} \right),$$

which is similar to P_1 and sends $\mathbb{B}^3$ into D_4^{IV}, is not equivalent to the rational isometry.

The map P is a counterexample to Conjecture 2.9 of Xiao–Yuan [47] for the case of maps from $\mathbb{B}^3$ into D_4^{IV}. Thus, the classification of proper holomorphic maps from $\mathbb{B}^3$ into D_4^{IV} seems to be even more subtle than that in higher dimensional case. We hope to return to this in the future.

References

[1] A.B. Aleksandrov, The existence of inner functions in a ball, Mat. Sb. (N.S.) 118(160) (2) (1982) 147–163, 287.

[2] H. Alexander, Proper holomorphic mappings in $\mathbb{C}^n$, Indiana Univ. Math. J. 26 (1) (1977) 137–146.

[3] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, Real Submanifolds in Complex Space and Their Mappings, vol. 47, Princeton University Press, 1999.

[4] V. Beloshapka, I. Kossovskiy, The sphere in $\mathbb{C}^2$ as a model surface for degenerate hypersurfaces in $\mathbb{C}^3$, Russ. J. Math. Phys. 22 (2015) 437–443.

[5] É. Cartan, Sur les domaines bornés homogènes de l'espace den variables complexes, Abh. Math. Semin. Univ. Hamb. 11 (1) (1935) 116–162.

[6] S.T. Chan, N. Mok, Holomorphic isometries of $\mathbb{B}^m$ into bounded symmetric domains arising from linear sections of minimal embeddings of their compact duals, Math. Z. 286 (1) (2017) 679–700.

[7] J.A. Cima, T.J. Suffridge, Proper holomorphic mappings from the two-ball to the three-ball, Trans. Am. Math. Soc. 311 (1) (1989) 227–239.

[8] J.A. Cima, T.J. Suffridge, Boundary behavior of rational proper maps, Duke Math. J. 60 (1) (1990) 135–138.

[9] J.P. D'Angelo, Proper holomorphic maps between balls of different dimensions, Mich. Math. J. 35 (1) (1988) 83–90.

[10] G. della Sala, B. Lamel, M. Reiter, Sufficient and necessary conditions for local rigidity of CR mappings and higher order infinitesimal deformations, Ark. Mat. 58 (2) (2020) 213–242.

[11] P. Ebenfelt, Normal forms and biholomorphic equivalence of real hypersurfaces in $\mathbb{C}^3$, Indiana Univ. Math. J. 47 (2) (1998) 311–366.

[12] J.J. Faran, Maps from the two-ball to the three-ball, Invent. Math. 68 (3) (1982) 441–475.

[13] J.J. Faran, The linearity of proper holomorphic maps between balls in the low codimension case, J. Differ. Geom. 24 (1) (1986) 15–17.

[14] G. Fels, W. Kaup, CR-manifolds of dimension 5: a Lie algebra approach, J. Reine Angew. Math. 604 (2007) 47–71.

[15] F. Forstnerič, Extending proper holomorphic mappings of positive codimension, Invent. Math. 95 (1) (1989) 31–61.

[16] J. Greilhuber, Smooth regularity of CR maps into boundaries of classical symmetric domains, Master's thesis, University of Vienna, 2020.

[17] M. Hakim, N. Sibony, Fonctions holomorphes bornées sur la boule unité de $\mathbb{C}^n$, Invent. Math. 67 (2) (1982) 213–222.

[18] H. Hamada, Rational proper holomorphic maps from $\mathbb{B}^n$ into $\mathbb{B}^{2n}$, Math. Ann. 331 (3) (2005) 693–711.

[19] L.-K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains, translated from the Russian by Leo Ebner and Adam Korányi, American Mathematical Society, Providence, R.I., 1963.

[20] X. Huang, On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions, J. Differ. Geom. 51 (1) (1999) 13–33.

[21] X. Huang, On a semi-rigidity property for holomorphic maps, Asian J. Math. 7 (4) (2003) 463–492.

[22] X. Huang, S. Ji, Mapping $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$, Invent. Math. 145 (2) (2001) 219–250.

[23] X. Huang, S. Ji, D. Xu, A new gap phenomenon for proper holomorphic mappings from B^n into B^N, Math. Res. Lett. 13 (4) (2006) 515–529.

[24] X. Huang, S. Ji, W. Yin, On the third gap for proper holomorphic maps between balls, Math. Ann. 358 (1–2) (2014) 115–142.

[25] X. Huang, J. Lu, X. Tang, M. Xiao, Boundary characterization of holomorphic isometric embeddings between indefinite hyperbolic spaces, Adv. Math. 374 (2020) 107388.

[26] S. Ji, A new proof for Faran's theorem on maps between $\mathbb{B}^2$ and $\mathbb{B}^3$, in: Recent Advances in Geometric Analysis, in: Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 101–127.

[27] W. Kaup, D. Zaitsev, On local CR-transformation of Levi-degenerate group orbits in compact Hermitian symmetric spaces, J. Eur. Math. Soc. (JEMS) 8 (3) (2006) 465–490.

[28] S.-Y. Kim, D. Zaitsev, Rigidity of proper holomorphic maps between bounded symmetric domains, Math. Ann. 362 (1) (2015) 639–677.

[29] M. Kolář, I. Kossovskiy, A complete normal form for everywhere Levi degenerate hypersurfaces in $\mathbb{C}^3$, arXiv preprint, arXiv:1905.05629, 2019.

[30] I. Kossovskiy, B. Lamel, M. Xiao, Regularity of CR-mappings of codimension one into Levi-degenerate hypersurfaces, Commun. Anal. Geom. 29 (1) (2021) 151–181.

[31] B. Lamel, Holomorphic maps of real submanifolds in complex spaces of different dimensions, Pac. J. Math. 201 (2) (2001) 357–387.

[32] B. Lamel, N. Mir, On the C^∞ regularity of CR mappings of positive codimension, Adv. Math. 335 (2018) 696–734.

[33] B. Lamel, N. Mir, Convergence and divergence of formal CR mappings, Acta Math. 220 (2) (2018) 367–406.

[34] B. Lamel, D.N. Son, The CR Ahlfors derivative and a new invariant for spherically equivalent CR maps, Ann. Inst. Fourier 71 (5) (2021) 2137–2167.

[35] E. Løw, A construction of inner functions on the unit ball in $\mathbb{C}^p$, Invent. Math. 67 (2) (1982) 223–229.

[36] E. Løw, Embeddings and proper holomorphic maps of strictly pseudoconvex domains into polydiscs and balls, Math. Z. 190 (3) (1985) 401–410.

[37] N. Mir, Holomorphic deformations of real-analytic CR maps and analytic regularity of CR mappings, J. Geom. Anal. 27 (3) (2017) 1920–1939.

[38] N. Mok, Geometry of holomorphic isometries and related maps between bounded domains, in: Geometry and Analysis, vol. 2, in: Adv. Lect. Math. (ALM), vol. 18, Int. Press, Somerville, MA, 2011, pp. 225–270.

[39] N. Mok, Some recent results on holomorphic isometries of the complex unit ball into bounded symmetric domains and related problems, in: Geometric Complex Analysis, in: Springer Proc. Math. Stat., vol. 246, Springer, Singapore, 2018, pp. 269–290.

[40] N. Mok, S.C. Ng, Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products, J. Reine Angew. Math. 669 (2012) 47–73.

[41] M.H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, Rend. Circ. Mat. Palermo (1884–1940) 23 (1) (1907) 185–220.

[42] M. Reiter, Classification of holomorphic mappings of hyperquadrics from $\mathbb{C}^2$ to $\mathbb{C}^3$, J. Geom. Anal. 26 (2) (2016) 1370–1414, arXiv:1409.5968.

[43] A.E. Tumanov, G.M. Khenkin, Local characterization of analytic automorphisms of classical domains, Dokl. Akad. Nauk SSSR 267 (4) (1982) 796–799.

[44] H. Upmeier, K. Wang, G. Zhang, Holomorphic isometries from the unit ball into symmetric domains, Int. Math. Res. Not. 2019 (1) (2019) 55–89.

[45] S.M. Webster, On mapping an n-ball into an $(n+1)$-ball in complex spaces, Pac. J. Math. 81 (1) (1979) 267–272.
[46] M. Xiao, Regularity of mappings into classical domains, Math. Ann. 378 (3–4) (2020) 1271–1309.
[47] M. Xiao, Y. Yuan, Holomorphic maps from the complex unit ball to type IV classical domains, J. Math. Pures Appl. 133 (2020) 139–166.